Exercises Bifurcation Theory Spring 2025

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1. Determine the generic 1-parameter steady-state bifurcation from the origin, in the class of C^∞ vector fields on $\mathbb{R}$ that **vanish at the origin**. In other words, let $f\colon \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be a one-parameter family of C^∞ vector fields on $\mathbb{R}$. Assume $f(0;\lambda) = 0$ for all $\lambda \in \mathbb{R}$ and $\frac{\partial f}{\partial x}(0;0) = 0$ and describe the generic structure of the zeroes of f around $(0;0)$. (Including possible stability data of the steady state points)

   You may use the following version of Hadamard's lemma: given a C^∞ map $f : \mathbb{R}^2 \mapsto \mathbb{R}$ such that $f(0, y) = 0$ for all $y \in \mathbb{R}$, there exists a C^∞ map $g : \mathbb{R}^2 \mapsto \mathbb{R}$ such that $f(x, y) = xg(x, y)$ for all $(x, y) \in \mathbb{R}^2$.

2. Determine the generic 1-parameter steady-state bifurcation from the origin, in the class of **odd** C^∞ vector fields on $\mathbb{R}$. In other words, let $f\colon \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be a one-parameter family of C^∞ vector fields on $\mathbb{R}$ satisfying $f(-x, \lambda) = -f(x, \lambda)$ for all $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. Assume (or deduce) that $f(0;0) = 0$ and assume $\frac{\partial f}{\partial x}(0;0) = 0$. Describe the generic structure of the zeroes of f around $(0;0)$. (Including possible stability data)

   You may again use Hadamard's, as well as what you know about (genericity of) the saddle-node bifurcation from class.

3. Let $F\colon \mathbb{R}^n \times \mathbb{R}^d \mapsto \mathbb{R}^n$ be a d-parameter family of C^∞ vector fields on $\mathbb{R}^n$. Suppose $F(0;0) = 0$ and write $A = D_xF(0;0)\colon \mathbb{R}^n \to \mathbb{R}^n$. Finally, let $U, V \subseteq \mathbb{R}^n$ be linear subspaces such that

   $$\mathbb{R}^n = U \oplus \ker(A) = V \oplus \operatorname{Im}(A)\,,$$

   where $\ker(A)$ and $\operatorname{Im}(A)$ are the kernel and image of A, respectively. Then Lyapunov-Schmidt reduction yields a locally defined map

   $$R\colon \ker(A) \times \mathbb{R}^d \to V\,.$$

   a. Assume first that F satisfies $F(-x, -\lambda) = F(x, \lambda)$ for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^d$. Show that R satisfies $R(-k, -\lambda) = R(k, \lambda)$ for all $k \in \ker(A)$ and $\lambda \in \mathbb{R}^d$.

   b. Now assume instead that there is a linear subspace $W \subseteq \mathbb{R}^d$ such that F satisfies $F(0, \lambda) = 0$ for all $\lambda \in W$. Show that R satisfies $R(0, \lambda) = 0$ for all $\lambda \in W$ as well.

4. Determine the center subspace and hyperbolic subspace of the following matrix
$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

5. In this exercise we consider the ODE on $\mathbb{R}^2$ given by:
$$\begin{aligned} \dot{x} &= x^3 + 4xy^2 \\ \dot{y} &= -y + 5x^2 + 6y^3 . \end{aligned}$$

   a. Determine the Taylor expansion around the origin, up to and including fifth order, of a function $\psi : W_c \to W_h$ whose graph is the centre manifold of the ODE above.

   b. Give the Taylor expansion around the origin, up to and including fifth order, of the reduced vector field $R : \mathbb{R} \to \mathbb{R}$ (where we identify W_c with $\mathbb{R}$). Argue in a single sentence whether the origin is stable or not in the above ODE on $\mathbb{R}^2$.

6. Given $k \geq 2$, we denote by $\mathcal{P}^k$ the linear space of all polynomial vector fields on $\mathbb{R}^2$ whose components are linear combinations of monomials of degree precisely k. Suppose we have the matrix
$$A = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} .$$
   Determine, for $k = 2, 3, \ldots, 7$ linear subspaces $V^k \subseteq \mathcal{P}^k$ such that
$$\mathcal{P}^k = V^k \oplus \operatorname{Im}(\operatorname{ad}_A |_{\mathcal{P}^k}) .$$
   Here $\operatorname{ad}_A |_{\mathcal{P}^k} : \mathcal{P}^k \to \mathcal{P}^k$ denotes the linear operation obtained by taking the Lie-bracket with the vector field $\xi(x) = Ax$, restricted to $\mathcal{P}^k$. Motivate your answer.