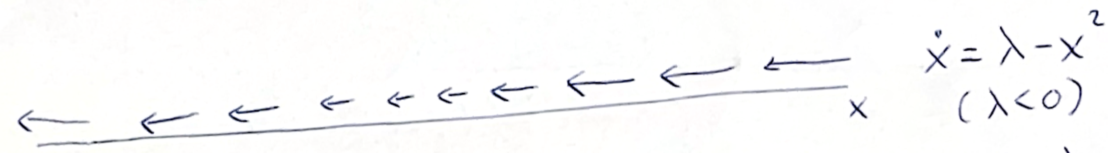
①

Ex 1 Consider, for each fixed value of $\lambda \in \mathbb{R}$, the vectorfield f_λ on $\mathbb{R}$, defined by

$$f_\lambda(x) = \lambda - x^2$$

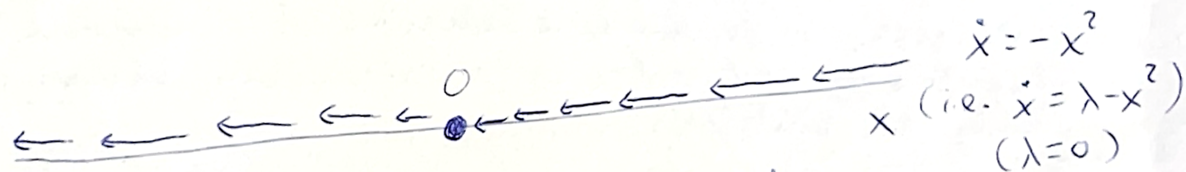
For $\lambda < 0$, a sketch of the phase portrait is given by

①



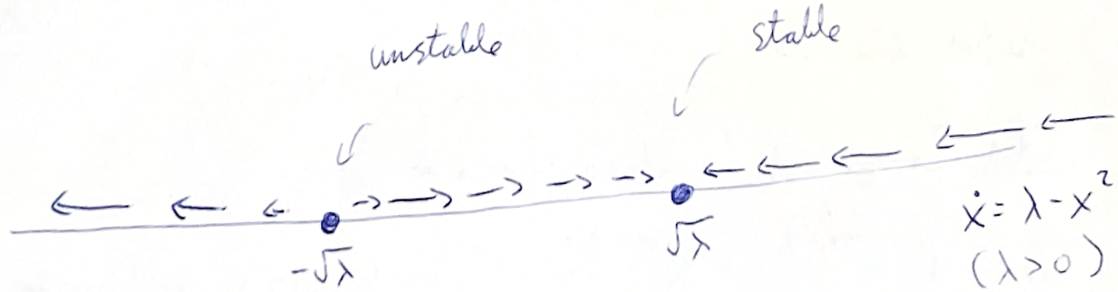
Though the exact formulas for the solutions vary in λ, the overal behavior, that is, the <u>qualitative</u> behavior, remains the same, as long as we keep λ (strictly) negative.
Things look different for $\lambda = 0$:

②



For instance, where for $\lambda < 0$ every solution "shot off" to $-\infty$, we now have a steady state point at $x = 0$. Moreover, solutions that start at a positive value of x limit to this steady-state point, and will of course not pass it.
Things are again different for $\lambda > 0$. Now we have 2 steady-state points, found by solving $f_\lambda(x) = \lambda - x^2 = 0$
So $x^2 = \lambda \Rightarrow x = \pm\sqrt{\lambda}$. The picture is now

③



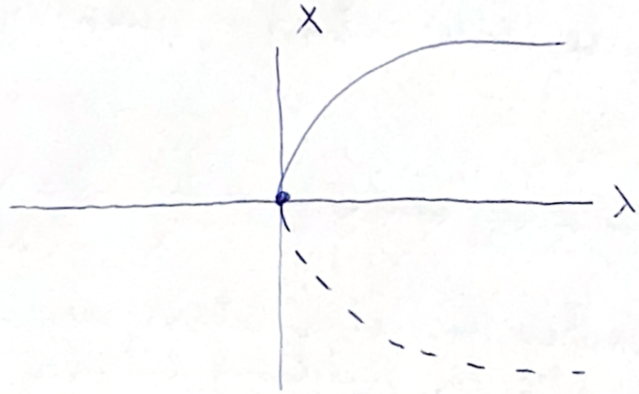
Note that $D_x f_\lambda(x) = -2x$, and so

$$D_x f_\lambda(x)\Big|_{x=-\sqrt{\lambda}} = +2\sqrt{\lambda} > 0$$

$$D_x f_\lambda(x)\Big|_{x=\sqrt{\lambda}} = -2\sqrt{\lambda} < 0$$

So $x = -\sqrt{\lambda}$ is unstable, and $x = \sqrt{\lambda}$ is stable

② We may, for each value of λ, plot the position of these fixed points, as well as their stability in the corresponding system on $\mathbb{R}$, $\dot{x} = f_\lambda(x)$:

④ 

Thus, each vertical slice (i.e. fixed value of λ) corresponds to one of the systems in the pictures ①, ② & ③. a solid line means the steady state point is stable in the corresponding ODE $\dot{x} = f_\lambda(x)$, a dashed line indicates it is unstable.
The picture ④ is referred to as a bifurcation diagram (of steady state points).

Next, we want to see what happens when we add higher order terms ("h.o.t") to the λ-family of ODES $\dot{x} = f_\lambda(x)$.
In other words, we consider

$$\dot{x} = \lambda - x^2 + a_{3,0} x^3 + a_{1,1} x \lambda + a_{0,2} \lambda^2 + \dots$$

for some $a_{ij} \in \mathbb{R}$. (so to speak)

To do so, we first need some notation and a very useful theorem:

<u>Thm</u>: <u>Implicit function theorem</u>

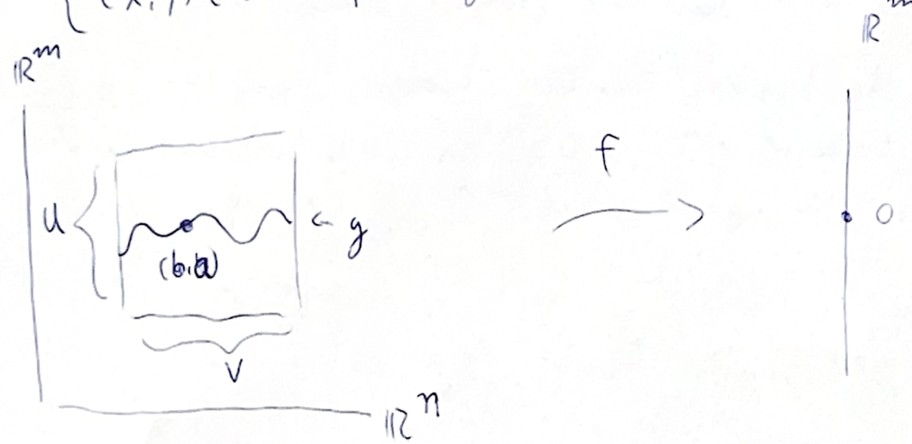
Let $f: \mathbb{R}^{m+n} \to \mathbb{R}^m$ be C^1 (continuously differentiable).
We write $f(x,y)$ with $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$. Suppose $f(a,b) = 0$ and that

$$D_x f(a,b) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a,b) & \cdots & \frac{\partial f_m}{\partial x_1}(a,b) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(a,b) & \cdots & \frac{\partial f_m}{\partial x_m}(a,b) \end{pmatrix}$$ is invertible

③ Then there exist open neighborhoods $U \subseteq \mathbb{R}^m$ s.t. $a \in U$,
$V \subseteq \mathbb{R}^n$ s.t. $b \in V$.

and a C^1 map $g : V \to U$ such that, as sets,

$$\{(x,y) \in U \times V \mid f(x,y) = 0\}$$
$$\shortparallel$$
$$\{(x,y) \in U \times V \mid x = g(y)\}$$



Note that, since $f(a,b) = 0$, we have $a = g(b)$.

- If f is C^k, for $k \in \{1,2,3,\ldots\} \cup \{\infty\}$, then g can also be assumed C^k

---

This theorem follows directly from:

Thm: <u>Inverse function theorem</u> (IFT)

Let $F : \mathbb{R}^l \to \mathbb{R}^l$ be C^1 and let $a \in \mathbb{R}^l$ be a point where $DF(a)$ is invertible (as linear map from $\mathbb{R}^l$ to $\mathbb{R}^l$) then there exist $U, V \subseteq \mathbb{R}^l$ open such that $a \in U$ and $F(U) \subseteq V$ and $H : V \to U$ a C^1 map such that

$$H(F(x)) = x \text{ for all } x \in U \quad \text{and}$$
$$F(H(y)) = y \text{ for all } y \in V.$$

So $F|_U$ is a bijection from U to V with C^1 inverse H

- If F is C^k, $k \in \{1,2,\ldots\} \cup \{\infty\}$, then H may be assumed C^k as well

The last part of the Inverse Function Theorem follows readily

Because, since $F(H(y)) = y$ for all $y \in V$, we have, by the chain rule.

$$DF(H(y)) \cdot DH(y) = Id.$$ or,

$$\text{So } DH(y) = (DF(H(y)))^{-1} = (\mathcal{I} \circ DF \circ H)(y)$$

where $\mathcal{I} : Gl(l;\mathbb{R}) \circlearrowleft$ is the C^∞ map that sends a matrix to its inverse

---

The Inverse Function Theorem is usually proven by finding a contraction on a suitable function space.

To obtain the Implicit Function Theorem from the Inverse Function Theorem, take $f : \mathbb{R}^{m+n} \to \mathbb{R}^m$ as in the Implicit function theorem and define $F : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ as $F(\underset{\in\mathbb{R}^m}{x}, \underset{\in\mathbb{R}^n}{y}) = (\underbrace{f(x,y)}_{\mathbb{R}^m}, \underbrace{y}_{\in\mathbb{R}^n})$

Then
$$DF(a,b) = \begin{pmatrix} D_x f(a,b) & D_y f(a,b) \\ 0 & Id|_{\mathbb{R}^n} \end{pmatrix}$$

So F satisfies the conditions of the Inverse function theorem
Let H be the local inverse, and write $H(x,y) = (\underset{\in\mathbb{R}^m}{H_1(x,y)}, \underset{\in\mathbb{R}^n}{H_2(x,y)})$
Then $F(H_1(x,y), H_2(x,y)) = (f(H_1(x,y), H_2(x,y)), H_2(x,y)) = (x,y)$

$$\text{So } H_2(x,y) = y \ \& \ f(H_1(x,y), y) = x$$

Now note that
$$\{(x,y) \mid f(x,y) = 0\} = \{(x,y) \mid F(x,y) = (0,y)\}$$
$$= \{H(0,y)\} = \{(H_1(0,y), y)\}$$

Set $g(y) := H_1(0,y)$ and we are done.

Next, we will often use "big $\mathcal{O}$" notation. Let $a \in \mathbb{R}^n$ and $f, g : \mathbb{R}^n \to \mathbb{R}$ two functions. We write $f(x) = \mathcal{O}(g(x))$ as $x \to a$ If there exist $C, \delta > 0$ such that

$$|f(x)| \le C|g(x)| \quad \text{for all } x \text{ s.t. } 0 < |x - a| < \delta$$

Most of the time, $g(x)$ will be something like $g(x) = |x^3|$ or $g(x) = |x^3| + |x||\lambda| + |\lambda|^2$

5

For instance, suppose $f(x,y) = x^2 f_1(x,y) + y^2 f_2(x,y)$ for some continuous functions $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$. Suppose $|f_1|$ and $|f_2|$ are bounded by $K>0$ in some neighbourhood $\mathcal{U}$ of $(0,0) \in \mathbb{R}^2$

then
$$|x^2 f_1(x,y) + y^2 f_2(x,y)| \leq |x^2||f_1(x,y)| + |y^2||f_2(x,y)|$$
$$\leq K(|x^2|+|y^2|) \quad \text{in } \mathcal{U}$$

So $f(x,y) = \mathcal{O}(|x^2|+|y^2|)$ as $x \to 0$.

Be careful with the "$|\cdot|$". For instance $x-y = \mathcal{O}(|x|+|y|)$ as $x\to 0$ as $|x-y| \leq |x|+|y|$. But $x-y \neq \underset{\text{as } x\to 0}{\mathcal{O}(x+y)}$, Because for $(x,y) = (\varepsilon, -\varepsilon)$.

$$|x-y|\Big|_{(x,y)=(\varepsilon,-\varepsilon)} = 2|\varepsilon| \nleq C\cdot|x+y|\Big|_{(x,y)=(\varepsilon,-\varepsilon)} = 0 \quad \text{in any neighborhood of } (0,0)$$

Recall Taylor's theorem:

If $f: \mathbb{R}^n \to \mathbb{R}$ is C^{k+1}, $k \geq 1$, then, for $a \in \mathbb{R}^n$,
$$f(x) = \sum_{|\alpha|\leq k} \frac{D^\alpha f(a)}{\alpha!}(x-a)^\alpha + \sum_{|\beta|=k+1} R_\beta(x)(x-a)^\beta$$
where $\alpha = (\alpha_1 \dots \alpha_n)$, $\beta = (\beta_1 \dots \beta_n)$ are multiindices, and every R_β is continuous (so locally bounded)

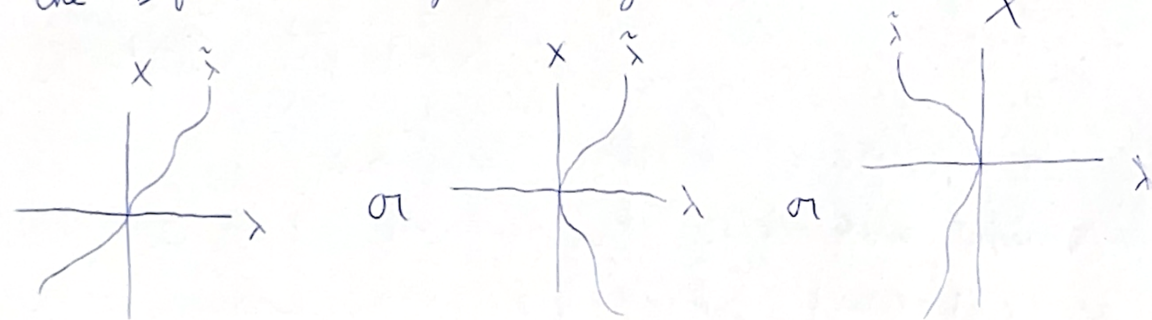
So then $f(x) = \sum_{|\alpha|\leq k} \frac{D^\alpha f(a)}{\alpha!}(x-a)^\alpha + \mathcal{O}\Big(\sum_{|\beta|=k+1} |x-a|^\beta\Big)$ as $x \to a$.

Typically, we will want to write $f(x) = \mathcal{O}(g(x))$ as $x\to 0$

So if we just write $f(x) = \mathcal{O}(g(x))$, we typically mean as $x \to 0$

Let us now study bifurcations in systems of the form
$$\dot{x} = \lambda - x^2 + \mathcal{O}(|x^3| + |x||\lambda| + |\lambda|^2) = f(x,\lambda)$$
(say the full system is C^∞, for simplicity)

⑥ Note that $f: \mathbb{R}^2 \to \mathbb{R}$ satisfies $D_\lambda f(0,0) = 1$. Thus by the Implicit function, there is a (locally defined) map $\tilde{\lambda}: \mathbb{R} \supseteq U \to V \subseteq \mathbb{R}$ satisfying $\tilde{\lambda}(0) = 0$ and such that all solutions to $f(x,\lambda) = 0$ are given by $(x, \tilde{\lambda}(x))$. This means the bifurcation diagram may look like, e.g.



(ignoring stability). All we know is that the λ-componts are a function of the x-components, of the steady-state points. Note that the 2nd and 3rd option would constitute "real bifurcations", as the number of zeroes changes as λ-varies.

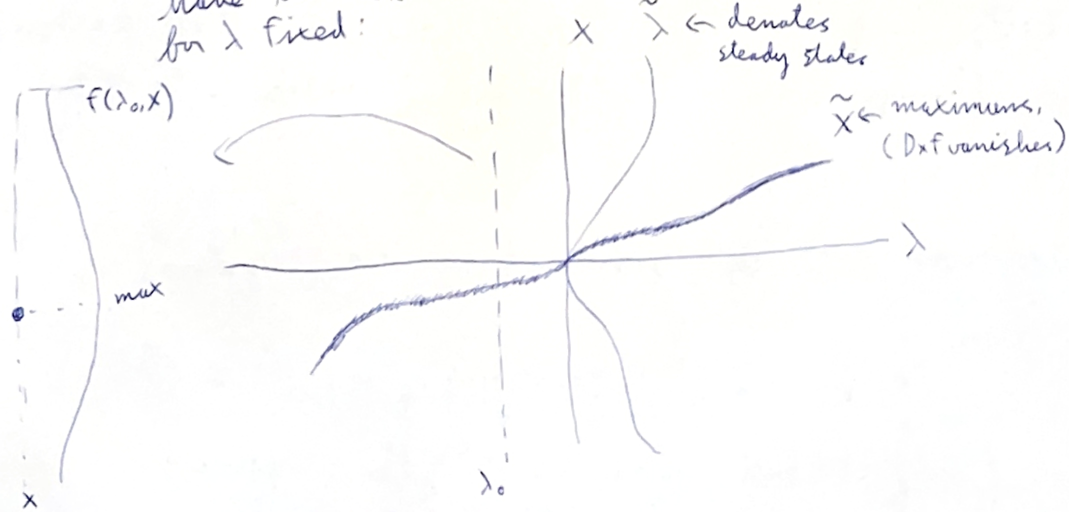
Now, the derivative in the x-direction of our family of vectorfields is given by $D_x f(x,\lambda) = -2x + \mathcal{O}(|x|^2 + |\lambda|)$. We see that $D_x f(0,0) = 0$ and, since $D_{xx} f(x,\lambda) = -2 + \mathcal{O}(|x| + |\lambda|)$, we conclude again by the Implicit function theorem that, locally,

$$\{(x,\lambda) \mid D_x f(x,\lambda) = 0\} = \{(\tilde{x}(\lambda), \lambda)\}$$

for some map $\tilde{x}$ passing through 0. Since $D_{xx} f(\tilde{x}(\lambda), \lambda) = -2 + \mathcal{O}(|\lambda|)$, we have

$$D_{xx} f(\tilde{x}(\lambda), \lambda) + 2 = \mathcal{O}(|\lambda|)$$

so locally $|D_{xx} f(\tilde{x}(\lambda), \lambda) + 2| \leq C|\lambda|$ for some $C > 0$. This means $D_{xx} f$ is negative along the curve $(\tilde{x}(\lambda), \lambda)$ around $(0,0)$. Thus we have that $(\tilde{x}(\lambda), \lambda)$ is a family of maximums of each $f(x,\lambda)$ for λ fixed:

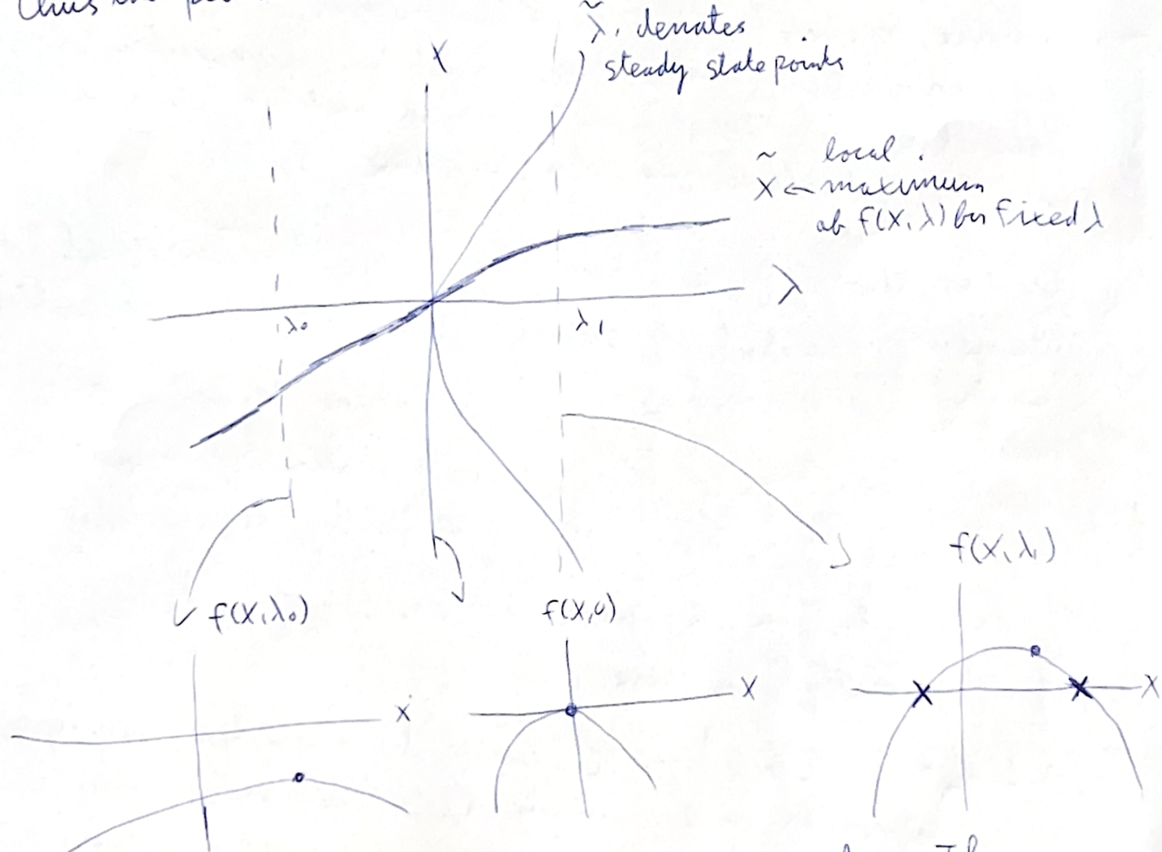


⑦ Finally, consider f along these maximums $\tilde{x}(\lambda)$.

Since $\tilde{x}(\lambda) = O(|\lambda|)$. and $f(x,\lambda) = \lambda - x^2 + O(|x|^3 + |x||\lambda| + |\lambda|^2)$, we have

$f(\tilde{x}(\lambda),\lambda) = \lambda + O(|\lambda|^2)$. So for $\lambda < 0$, $f(\tilde{x}(\lambda),\lambda) < 0$
for $\lambda > 0$ $f(\tilde{x}(\lambda),\lambda) > 0$.

So for $\lambda_0 < 0$, the maximum of $f(x,\lambda_0)$ when varying x is negative, so we cannot have any zeroes (locally) for $\lambda < 0$.
Thus the picture must be



Note that we may also deduce stability from this. Thus, qualitatively, the bifurcation diagrams of $\dot{x} = \lambda - x^2$ and $\dot{x} = \lambda - x^2 + O(|x|^3 + |x||\lambda| + |\lambda|^2)$ are the same. The name for this common steady-state bifurcation is the <u>saddle-node</u> bifurcation.

This is what the vectorfield $\dot{x} = \lambda - x^2 + O(|x|^3 + |x||\lambda| + |\lambda|^2)$ "means," by the way

$$
\begin{aligned}
f(x,\lambda) = \; & a_{00} + a_{10}x + a_{20}x^2 + a_{30}x^3 + \dots \\
& + a_{01}\lambda + a_{11}x\lambda + a_{21}x^2\lambda + a_{31}x^3\lambda + \dots \\
& + a_{02}\lambda^2 + a_{12}x\lambda^2 + a_{22}x^2\lambda^2 + a_{32}x^3\lambda^2 + \dots \\
& + a_{03}\lambda^3 + a_{13}x\lambda^3 + \dots
\end{aligned}
$$

← $O(|x|^3)$

$O(|x||\lambda|)$

$O(|\lambda|^2)$

So $a_{00} = 0$, so $f(0,0) = 0$
$a_{10} = 0$ so $D_x f(0,0) = 0$
$a_{20} = -1$
$a_{01} = 1$

(8) It seems a bit excessive that we need 4 conditions for this to hold, but as it turns out this situation is "generic", as we now explore.

First, note that, if $f(x_0, \lambda_0) = 0$ for some $x_0, \lambda_0 \in \mathbb{R}$, then we may define $\tilde{f}(x, \lambda) := f(x + x_0, \lambda + \lambda_0)$, so that $\tilde{f}(0,0) = 0$. Thus, after a quick coordinate transformation, we may assume that the steady-state point that we want to investigate, in terms of continuation, is at $x = 0$ for $\lambda_0 = 0$. This of course works for vectorfields on $\mathbb{R}^n$ in d-paramaters as well. One word of advice, going from f to $\tilde{f}$ as above may not preserve all the properties that f has, such as symmetry or a network-structure, at least some care is needed. This explains why in our situation, we may assume $a_{00} = 0$ (i.e. $f(0,0) = 0$)

As for a_{10}, if $a_{10} \neq 0$, then by the Implicit Function Theorem, the set of steady state points is locally a graph over λ

So there is not really a bifurcation to speak of (of steady state points)

This is why we assume $a_{10} = 0$.

More generally, we have:

<u>Thm 1</u> Let $F: \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ be a d-paramater family of vectorfields on $\mathbb{R}^n$, which is C^1 (seen as a map from $\mathbb{R}^{n+d} \to \mathbb{R}^n$). Suppose $F(0,0) = 0$, and that $x = 0$ is a hyperbolic steady state point of $F_0 := F(\bullet, 0): \mathbb{R}^n \to \mathbb{R}^n$. Then there exist open neighbourhoods $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^d$ with $0 \in U$, $0 \in V$ and a C^1 map $g: V \to U$ such that

1) The steady-state points of F are precisely given by $\{(x, \lambda) \in U \times V \mid x = g(\lambda)\}$

2) all the steady state points in $U \times V$ are hyperbolic for their respective systems $F_\lambda := F(\bullet, \lambda)$ on $\mathbb{R}^n$. i.e. each matrix $D_x F(g(\lambda); \lambda)$ is <u>hyperbolic</u>

⑨ The same statement holds true if we replace "hyperbolic" in the theorem above with "invertible", in both instances

—

Recall that a matrix is hyperbolic if it has no eigenvalues on the imaginary axis, and invertible if it has no 0-eigenvalues.
Thm. 1 essentially tells us that hyperbolic steady-states persist.
The following Lemma will be extremely useful:

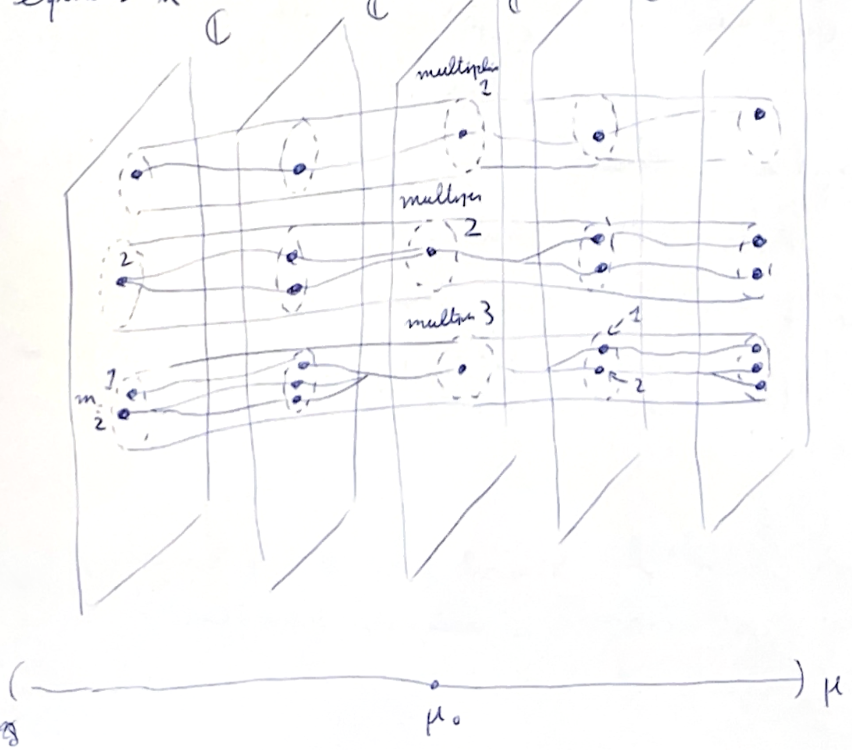
<u>Lem 2</u> Let $A: \mathbb{R}^d \to Mat(n, \mathbb{C})$ be a continuous family of complex $n \times n$ matrices. Then the eigenvalues are continuous in the following way: Given μ^0 (and so A_{μ_0}), given an $\varepsilon > 0$, there is a $\delta > 0$ such that if $\|\mu - \mu_0\| < \delta$ then the eigenvalues of A_μ lie in the union of open balls

$$B_\varepsilon(\lambda_1) \cup B_\varepsilon(\lambda_2) \cup \dots \cup B_\varepsilon(\lambda_n)$$, where $\lambda_1, \dots, \lambda_n$ are the (not necessarily distinct) eigenvalues of A_{μ_0}

In fact: Given μ_0, given ε and given λ an eigenvalue of A_{μ_0} with algebraic multiplicity k, there is a $\delta > 0$ such that if $\|\mu - \mu_0\| < \delta$, then the sum of the algebraic multiplicities of the eigenvalues of A_μ that lie in $B_\varepsilon(\lambda)$ equals k

ε small enough [such that $B_\varepsilon(\lambda)$ contains as eigen values of A_{μ_0} only λ]

eg



(10) Proof of Lemma 2 : Denote by

$$p_{X,\mu}(\cancel{\lambda,x}) = X^n + a_{n-1}(\mu) X^{n-1} + a_{n-2}(\mu) X^{n-2} + \dots + a_1(\mu) X + a_0(\mu)$$

$$= \det(X \mathrm{Id}_n - A_\mu)$$

Suppose μ_0, $\varepsilon > 0$ and λ an eigenvalue of A_{μ_0} are given.

Denote by $q(X,\mu) = \frac{\partial p}{\partial x}(X,\mu)$, which is not the 0-polynomial.

Recall ~~...~~ that the only zero of $p(X,\mu)$ that is contained in $\overline{B_{\varepsilon}(\lambda)}$ is λ itself.

Then by the argument principle,

$$\frac{1}{2\pi i} \oint_C \frac{q(z,\mu_0)}{p(z,\mu_0)}\, dz = \text{" \# zeroes of } p_{\mu_0} \text{ in } B_\varepsilon(\lambda), \text{ counted with multiplicity"}$$

$$= k$$

where C denotes a circle of radius ε, integrated counter-clockwise, around λ.

Then C has on it no zeroes of $p(z,\mu_0)$. So (since the circle is compact) there exists ~~no~~ a $\delta_1 > 0$ such that $p(z,\mu)$ does not vanish on C if $\|\mu - \mu_0\| < \delta_1$.

So the map $\Psi : B_{\delta_1}(\mu_0) \subseteq \mathbb{R}^d \longrightarrow \mathbb{C}$

$$\mu \longmapsto \frac{1}{2\pi i} \oint_C \frac{q(z,\mu)}{p(z,\mu)}\, dz$$ is well-defined.

It is also continuous. Now choose $\delta_1 > \delta > 0$ such that, if

$\|\mu - \mu_0\| < \delta$ then $\|\Psi(\mu) - k\| < \frac{1}{2}$.

Since Ψ takes on integer values, necessarily

$$\Psi(\mu) = \frac{1}{2\pi i} \oint_C \frac{q(z,\mu)}{p(z,\mu)}\, dz = \text{" \# zeroes of } p_\mu \text{ in } B_\varepsilon(\lambda) \text{ counted with multiplicity"}$$

$$= k$$

This proves the second part of the Lemma.
The first follows immediately from that

 Note that Thm 1 now follows immediately from Lemma 1 and the Implicit function theorem

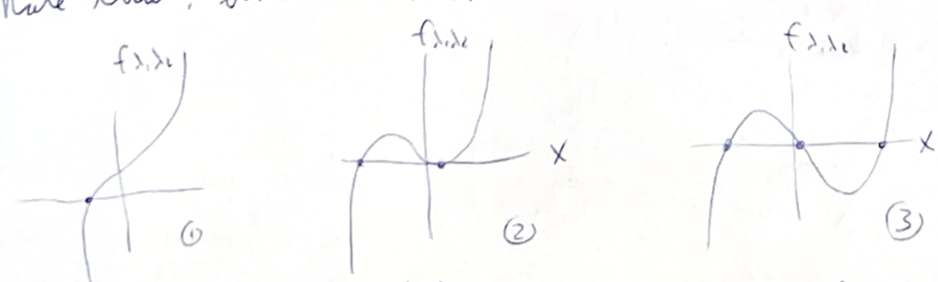
To further drive home that we are not interested in hyperbolic fixed points when doing bifurcation analysis (at least local bifurcation analysis), recall that

<u>Thm</u> <u>Hartman-Grobman</u>: Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vectorfield and p a hyperbolic fixed point of the flow of F, so $F(p)=0$ and $DF(p) \in Mat(n;\mathbb{R})$ is hyperbolic. Then there exist open neighborhoods $p \in U \subseteq \mathbb{R}^n$, $0 \in V \subseteq \mathbb{R}^n$ and a <u>homeomorphism</u> $H: U \to V$ which conjugates the restriction of $\dot{x} = F(x)$ on U to that of $\dot{y} = DF(p)y$ on V. (That is, if $\varphi_t(x)$ denotes the flow of $\dot{x} = f(x)$ and $\psi_t(y)$ that of $\dot{y} = DF(p)y$. Then $\varphi_t(x)$ is defined iff $\psi_t(H(x))$ is defined, and then $H\varphi_t(x) = \psi_t(H(x))$)

So far we have only considered 1 parameter bifurcations, but others exist as well, of course. Consider e.g.

$$\dot{x} = f_{\lambda_1\lambda_2}(x) = x^3 - \lambda_1 x + \lambda_2 \qquad , x, \lambda_1, \lambda_2 \in \mathbb{R}$$

Note that, for fixed $\lambda_1 \lambda_2$, $f_{\lambda_1\lambda_2}(x) = 0$ has 1, 2 or 3 solutions:



The transition between states ① and ③ is ②, where we have a double zero. There: $f_{\lambda_1\lambda_2}(x) = x^3 - \lambda_1 x + \lambda_2 = 0$ ☆

and $\dfrac{\partial_x f_{\lambda_1\lambda_2}(x)}{\partial x} = 3x^2 - \lambda_1 = 0$ ☆☆

So $\underline{\lambda_1 = 3x^2}$, from ☆☆. Then from ☆

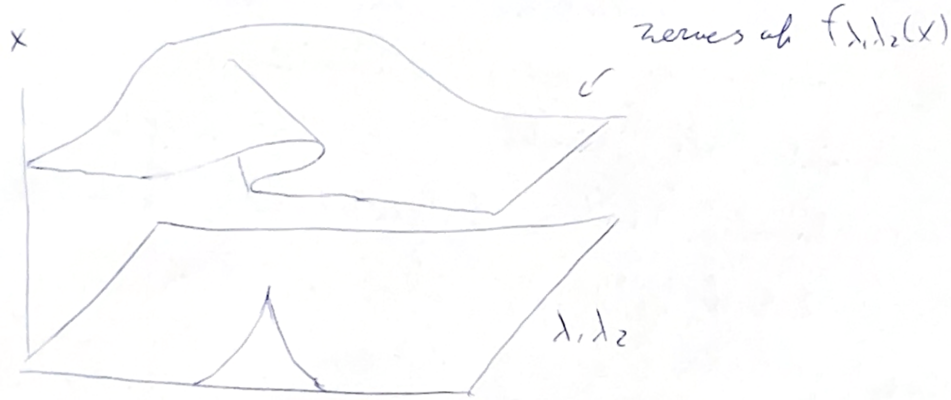
$$0 = x^3 - \lambda_1 x + \lambda_2 = x^3 - 3x^2 \cdot x + \lambda_2$$
$$= x^3 - 3x^3 + \lambda_2 = -2x^3 + \lambda_2$$

So $\underline{\lambda_2 = 2x^3}$

(12) Together, these give

$$\left.\begin{aligned} &4\lambda_1^3 - 27\lambda_2^2 = \\ &2^2 \cdot 3^3 x^6 - 3^3 2^2 x^6 = 0 \end{aligned}\right\} \text{ so } 4\lambda_1^3 = 27\lambda_2^2$$

This is a cusp, and yields the famous cusp-catastrophe:



Finally, I want to show a bifurcation where the number of steady-state points does not change at all (though the dynamical behavior does change, significantly!).

Consider the system on $\mathbb{R}^2$, parametrised by $\lambda \in \mathbb{R}$;

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \lambda x - \omega y - (x^2+y^2)x \\ \omega x + \lambda y - (x^2+y^2)y \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - (x^2+y^2)\begin{pmatrix} x \\ y \end{pmatrix}$$

, where $\omega \in \mathbb{R}_0$ ($\omega > 0$) is a fixed constant (not a variable, nor a bifurcation parameter).

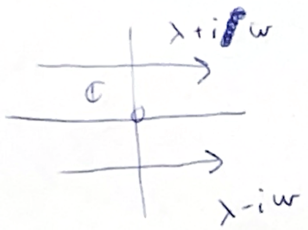
$$= F_\lambda(x, y)$$

Note that the eigenvalues of

$D_x F_\lambda(0,0) = \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix}$ are given by $\lambda \pm i\omega$

Thus, $D_x F_\lambda(0,0)$ is always invertible; the isolated steady state point at $(0,0)$ persists. It is in fact the only steady state point of the system, for all values of $\lambda \in \mathbb{R}$.

(13) However, at $\lambda = 0$, the system looses hyperbolicity. In fact, as λ moves from negative to positive, the eigenvalues move as



To see what happens, we write the system as one on $\mathbb{C}$ by setting $z = x+iy$. we get

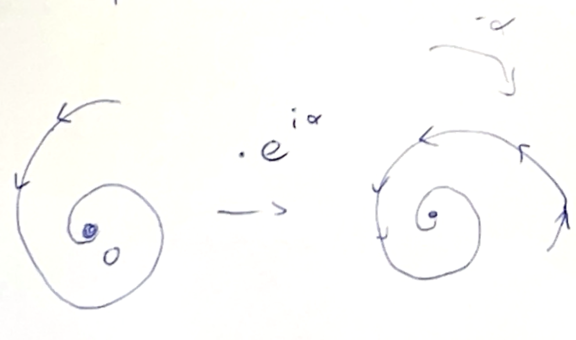
$$\dot{z} = (\lambda + i\omega) z - |z|^2 z \qquad ☺$$

This already shows one peculiarity in the system: Suppose $z(t)$ is a solution, and consider the curve $w(t) = e^{i\alpha} z(t)$, for some $\alpha \in \mathbb{R}$.

then
$$\begin{aligned}\dot{w}(t) = e^{i\alpha}\dot{z}(t) &= e^{i\alpha}(\lambda+i\omega) z(t) - e^{i\alpha}|z(t)|^2 z(t) \\ &= (\lambda + i\omega) e^{i\alpha} z(t) - z(t)\cdot\overline{z(t)}\cdot e^{i\alpha} z(t) \\ &= (\lambda+i\omega) e^{i\alpha} z(t) - (e^{i\alpha} z(t))(e^{-i\alpha}\overline{z(t)}) e^{i\alpha} z(t) \\ &= (\lambda + i\omega) w(t) - |w(t)|^2 w(t)\end{aligned}$$

as $\boxed{e^{i\alpha}\cdot e^{-i\alpha} = 1}$

Thus $w(t) := e^{i\alpha} z(t)$ satisfies ☺ whenever $z(t)$ does. The family of linear maps $\{e^{i\alpha} \mid \alpha \in \mathbb{R}\}$ "represents the Lie Group S^1" (More on this later), and we say that the Equation ☺ has a symmetry: There are linear maps sending solutions to solutions



Let us now write a solution of ☺ in polar coordinates:

$$z(t) = r(t)\, e^{i\theta(t)}.$$

$z(t) \equiv 0$ is always a solution, so if we choose any other solution, this is OK.

(14) Differentiating, we get

$$\dot{z} = \dot{r}e^{i\theta} + i\dot{\theta}re^{i\theta} \quad \text{so}$$

$$\bar{z}\dot{z} = \dot{r}e^{i\theta}re^{-i\theta} + i\dot{\theta}re^{i\theta}re^{-i\theta}$$

$$= \dot{r}r + i\dot{\theta}r^2$$

Thus

$$\dot{r} = \mathrm{Re}\left(\frac{\bar{z}\dot{z}}{r}\right) = \frac{1}{r}\mathrm{Re}(\bar{z}\dot{z}) \quad , \quad \dot{\theta} = \mathrm{Im}\left(\frac{\bar{z}\dot{z}}{\|z\|^2}\right) = \frac{1}{r^2}\mathrm{Im}(\bar{z}\dot{z})$$

Now $\bar{z}\dot{z} = (i\omega+\lambda)z\bar{z} + -|z|^2 z\bar{z}$

$$= (i\omega+\lambda)\|z\|^2 - |z|^4$$

$$= (i\omega+\lambda)r^2 - r^4$$

Thus $\dot{r} = \frac{1}{r}\mathrm{Re}((i\omega+\lambda)r^2 - r^4) = \frac{1}{r}(\lambda r^2 - r^4)$

$$= \lambda r - r^3$$

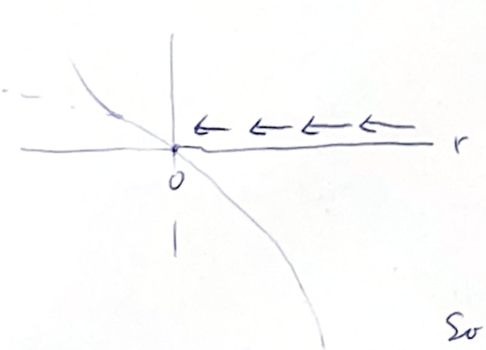
and $\dot{\theta} = \frac{1}{r^2}\mathrm{Im}((i\omega+\lambda)r^2 - r^4) = \frac{1}{r^2}\omega r^2 = \omega$

So always $\dot{\theta} = \omega \Rightarrow \theta = \omega t + \theta_0$

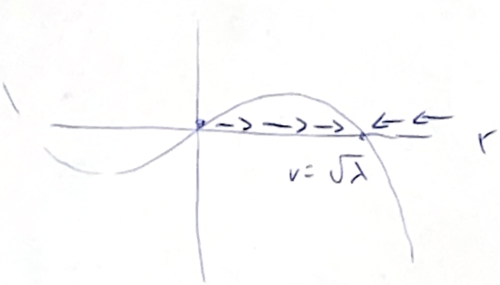
and $\dot{r} = \lambda r - r^3$

Note that $\lambda r - r^3 = 0$ gives $r = 0$ and $r = \pm\sqrt{\lambda}$, if $\lambda \geq 0$:

$\lambda r - r^3 \quad \lambda \leq 0$



$\lambda r - r^3 \ , \ \lambda > 0.$



So for $\lambda > 0$, we get the solutions

$$z(t) = \sqrt{\lambda}\, e^{i(\omega t + \theta_0)}$$

which "are" periodic (it's one periodic solution, with differing $z(0)$, of course)

(15) We get the famous Hopf - Bifurcation. for $\lambda \leq 0$, we have a stable steady state point, as $\lambda > 0$, it looses stability, but a stable periodic orbit emerges

Again, the bifurcation persists when higher order terms are added.

- Other "famous bifurcation" to keep an eye on:

- Pitchfork

- Transcritical, i.e.

(16) # Part 2 : Reduction Techniques

We now describe so-called "(complexity) reduction techniques". Most notably dimension reduction techniques. We start with Lyapunov-Schmidt reduction. To illustrate it, consider the 1-parameter family of ODEs on $\mathbb{R}^2$ given by

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x + y + 3\lambda + xy \\ \lambda - xy \end{pmatrix} =: F_\lambda(x, y)$$

Note that $F_0(0,0) = 0$ and $DF_0(0,0) = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$, the latter of which is not hyperbolic, so we may not assume this steady state point persists. To solve $F_\lambda(x,y) = 0$ near $\lambda = 0$, $x = y = 0$, we need to solve

$$F^1_\lambda(x,y) = -x + y + 3\lambda + xy = 0 \quad (1) \quad \&$$
$$F^2_\lambda(x,y) = \lambda - xy = 0 \quad (2).$$

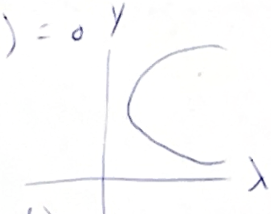
From the Implicit Function Theorem, as $D_x F^1_\lambda(0,0) = -1 + y|_{y=0} = -1$ we see that (1) is locally solved by $x = x(\lambda, y)$.
We write $x(y, \lambda) = ay + b\lambda + \mathcal{O}(|y|^2 + |y||\lambda| + |\lambda|^2)$, then plugging into (1) we get

$$-x(y,\lambda) + y + 3\lambda + x(y,\lambda)y = 0$$
$$-ay - b\lambda + y + 3\lambda + \mathcal{O}(|y|^2 + |y||\lambda| + |\lambda|^2) = 0$$
$$\Rightarrow a = 1,\ b = 3, \text{ so}$$
$$x(y,\lambda) = y + 3\lambda + \mathcal{O}(|y|^2 + |y||\lambda| + |\lambda|^2)$$

Now we plug this equation for $x(y,\lambda)$ into (2), to obtain

$$\lambda - x(y,\lambda)\, y = 0 \Rightarrow$$
$$\lambda - y^2 - 3\lambda y + \mathcal{O}(|y|^3 + |y|^2|\lambda| + |y||\lambda|^2) = 0$$
$$\Rightarrow \lambda - y^2 + \mathcal{O}(|y|^3 + |\lambda||y|) = 0$$

So the steady states look like



(with the equations for x determined by λ and λ)

(17) So what did we do here? This is an instance of so-called <u>Lyapunov-Schmidt Reduction</u>. Suppose we have a family of vectorfields (C^∞) $F: \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ s.t. $F(0;0)=0$. to solve for $F(x;\lambda)=0$ locally around $(x;\lambda)=(0,0)$, let us write $A := D_x F(0;0)$. We choose a vectorspace $U \subseteq \mathbb{R}^n$ complementary to $\ker(A)$ and a vectorspace $V \subseteq \mathbb{R}^n$ complementary to $\mathrm{Im}(A)$. Thus

$$\mathbb{R}^n = U \oplus \ker(A) = V \oplus \mathrm{Im}(A).$$

Denote by

$P_V : \mathbb{R}^n \to V$ the projection onto V along $\mathrm{Im}(A)$. So

$Id_V - P_V : \mathbb{R}^n \to \mathrm{Im}(A)$ is the projection onto $\mathrm{Im}(A)$ along V.

Then $F(x;\lambda)=0$ is equivalent to

$$(Id_V - P_V)F(x;\lambda) = (Id_V - P_V)F(u+y;\lambda) = 0 \quad (1)$$

&

$$P_V F(x;\lambda) = P_V F(u+y;\lambda) = 0 \quad (2)$$

where we write $x = u+y$ with $u \in U$, $y \in \ker(A)$.

We first focus on Equation (1): Define $G: U \times \ker(A) \times \mathbb{R}^d \to \mathrm{Im}(A)$ by $G(u,y;\lambda) = (Id_V - P_V)(F(u+y;\lambda))$

given $p \in U$, we have $D_u G(0,0;0)p = \frac{d}{d\epsilon}\big|_{\epsilon=0} G(\epsilon p, 0;0)$

$$= \frac{d}{d\epsilon}\Big|_{\epsilon=0}(Id_V - P_V)F(\epsilon p;0) = (Id_V - P_V)D_x F(0;0)p$$
$$= (Id_V - P_V)Ap = Ap.$$

where in the last step we use that $Ap \in \mathrm{Im}(A)$, so $(Id_V - P_V)Ap = Ap$

Now, if $D_u G(0,0;0)p = 0$ then $Ap = 0$ so $p \in \ker A$, but $p \in U$ and $\mathbb{R}^n = U \oplus \ker(A)$ so $p = 0$. Thus $D_u G(0,0;0)$ is injective.

By <u>rank-nullity</u> $\dim \ker(A) + \dim(\mathrm{Im}(A)) = \dim(\mathbb{R}^n) = n$

(As $0 \to \ker(A) \xrightarrow{i} \mathbb{R}^n \xrightarrow{\text{"}F\text{"}} \mathrm{Im}(A) \to 0$ is exact)

Also $\dim(U) + \dim(\ker(A)) = n$ so $\dim(\mathrm{Im}(A)) = \dim(U)$. Thus $D_u G(0,0;0) : U \to \mathrm{Im}(A)$ is <u>bijective</u>. So by the Implicit Function theory: (1) is solved locally by $(u(y;\lambda), y;\lambda)$

Now define $H: \ker(A) \times \mathbb{R}^d \to V$ by

$$H(y;\lambda) = P_V F(u(y;\lambda)+y;\lambda)$$

$\boxed{(3): H(y;\lambda) = 0}$

Then solutions to (3) are the same as solutions to (2), given (1) is satisfied.

18: Note as well that

$$\dim(V) = n - \dim(\operatorname{Im}(A)) = \dim(\ker(A)).$$

So the only place we have to get "our hands dirty" is at solving $H(y;\lambda) = 0$ for $y \in \ker(A)$.

<u>Example</u> Consider the ODE:

$$\begin{aligned}
\dot{x}_1 &= -x_1 + \sin(x_1) y^2 \\
\dot{x}_2 &= -x_2 + 2\sin(x_2) y^2 \\
\dot{x}_3 &= -x_3 + 3\sin(x_3) y^2 \\
&\vdots \\
\dot{x}_{100} &= -x_{100} + 100 \sin(x_{100}) y^2 \\
\dot{y} &= -\lambda + y^2 + x_1^3 + x_2^3 + \dots + x_{100}^3
\end{aligned} \quad = f_\lambda(x_1 \dots x_{100}, y)$$

with $x_1, x_2 \dots x_{100}, y, \lambda \in \mathbb{R}$. What is the structure of the zeroes near $(0,0 \dots 0,0;0) \in \mathbb{R}^{102}$?

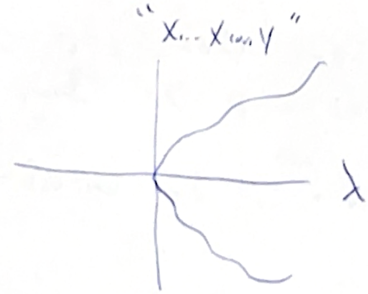
By the <u>Implicit Function Theorem</u>,☆ $x_i = x_i(y;\lambda) = \mathcal{O}(|y|+|\lambda|)$

Then to solve for the y-variable:

$$0 = -\lambda + y^2 + x_1^2 + \dots + x_{100}^2 = -\lambda + y^2 + \mathcal{O}(|y|^3 + |\lambda|^2|y| + |\lambda||y|^2 + |\lambda|^3)$$

Thus we have to solve $\lambda - y^2 + \mathcal{O}(|y|^3 + |\lambda||y| + |\lambda|^2) = 0$

So again we get



locally

The Lyapunov - Schmidt Reduction technique is more of a "big deal" then you might think: as we have:

Theorem (informally): Let $U \subseteq \mathbb{R}^d$ open, bounded, and let $C(U;n)$ denote all smooth maps $l : U \to \operatorname{Mat}(\mathbb{R};n)$. Then for "generic choice" of $l \in C(U;n)$, we have

$$\dim(\ker l(x)) \leq d \text{ for all } x \in U$$

(19) Thus, in particular, along a 1-paramater bifurcation (so $d=1$) we have generically that $\dim(\ker(A))=1$. So then the reduced equation $H(y;\lambda)$ just becomes a problem on $\mathbb{R}$ (with 1-paramater) (even if $n=10^7$!) More on this later.
Later, I also want to show Lyapunov-Schmidt Reduction on Banach spaces.
Note that one problem with Lyapunov-Schmidt reduction is that we do not (easily) get information on Stability. This is Different with our next technique:

Center Manifold Reduction

To motivate it, we need the following result & definition:

Lem Let $A \in \mathrm{Mat}(\mathbb{R};n)$ be a given real $n\times n$ matrix, and denote by $\Lambda_1, \Lambda_2 \dots \Lambda_l \subseteq \mathbb{R}$ a partition of $\mathbb{R}$ into l parts. So $\Lambda_i \cap \Lambda_j = \emptyset$ if $i\neq j$ $(i,j \in \{1..l\})$ and $\mathbb{R} = \Lambda_1 \cup \Lambda_2 \cup \cup \Lambda_l$
Then there exist unique linear subspaces $U_1, U_2 .. U_l \subseteq \mathbb{R}^n$ s.t.

1) $AU_i \subseteq U_i$ for all $i \in \{1,..,l\}$
2) $A|_{U_i}: U_i \to U_i$ has only (generalised) eigenvalues with real parts in Λ_i, for all $i\in\{1..,l\}$
3) $U_1 \oplus U_2 \oplus \quad \oplus U_l = \mathbb{R}^n$

Proof This follows directly from real Jordan-Normal form.
Note that, if λ is an eigenvalue of A with $\mathrm{Re}(\lambda) \in \Lambda_i$ then $\bar{\lambda}$ is an eigenvalue, satisfying likewise $\mathrm{Re}(\bar{\lambda}) \in \Lambda_i$

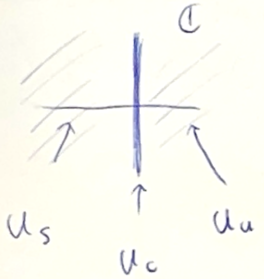
(as $\mathrm{Re}(\lambda) = \mathrm{Re}(\bar{\lambda})$)

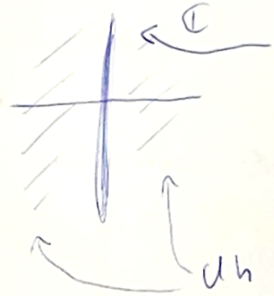
Definition Two cases are important to us:
Case 1) $\Lambda_1 = \{x\in\mathbb{R} \mid x<0\}$, $\Lambda_2 = \{0\}$, $\Lambda_3 = \{x\in\mathbb{R} \mid x>0\}$
then U_1 is called the stable subspace of A, we write $U_1 = U_s$
U_2 is called the center subspace of A, we write $U_2 = U_c$
U_3 is called the unstable subspace of A, we write $U_3 = U_u$

Case 2) $\Lambda_1 = \{x \in \mathbb{R} \mid x \neq 0\}$, $\Lambda_2 = \{0\}$ then

- U_1 is the hyperbolic subspace of A, $U_1 = U_h$
  U_2 is the center subspace of A, $U_2 = U_c$ (as before).

Note that $U_h = U_s \oplus U_u$, and so $\mathbb{R}^n = U_h \oplus U_c = U_s \oplus U_u \oplus U_c$





(Of course U_c is the same in both definitions)

We note the following: dynamics on U_u go to infinity as $t \to \infty$ while those for U_s to go infinity as $t \to -\infty$. That is, dynamics of the system $\dot{x} = Ax$, so $e^{tA}x_0$ for $x_0 \in U_u, U_s$ respectively. So the bounded solutions, if present, have to lie on U_c. Of course these are not too interesting for linear systems, but we can have families of steady state points or periodic orbits, and these have to lie on U_c. We explore dynamics of $\dot{x} = Ax$ some more in the following:

<u>Prop 1</u>: Denote by π_c & $\pi_h = Id - \pi_c$ the projections along $\mathbb{R}^n = U_h \oplus U_c$ onto U_c and U_h, resp.
Similarly, define π_c, π_u, π_s w.r.t. $\mathbb{R}^n = U_c \oplus U_u \oplus U_s$

Set $\beta_+ := \min\{Re(\lambda) \mid \lambda$ eigenvalue of A and $\lambda > 0\}$
$\beta_- := \max\{Re(\lambda) \mid \lambda$ " " of A and $\lambda < 0\}$

Then given $\varepsilon > 0$, there exists $M(\varepsilon) > 0$ s.t.

$$\| e^{At}\pi_c \| \leq M(\varepsilon)\, e^{\varepsilon|t|} \quad \forall t \in \mathbb{R}$$

$$\| e^{At}\pi_u \| \leq M(\varepsilon)\, e^{(\beta_+ - \varepsilon)t} \quad \forall t \leq 0$$

$$\| e^{At}\pi_s \| \leq M(\varepsilon)\, e^{(\beta_- + \varepsilon)t} \quad \forall t \geq 0$$

— Thus the U_c part may still grow, but less than any given exponential.

The stable part will decrease, and do better than "the highest eigenvalue plus a bit"

21) The proof of Prop 1 uses the so-called Jordan - Chevalley decomposition. It states, given $A \in \mathrm{Mat}(n;\mathbb{R})$, there exist unique $S, N \in \mathrm{Mat}(n;\mathbb{R})$, satisfying

1) $A = S + N$
2) S is semisimple (diagonizable in $\mathrm{Mat}(n;\mathbb{C})$)
3) N is nilpotent ($N^k = 0$ for some $k > 0$)
4) $SN = NS$

Note that, if A has the Jordan-normal form

$$BAB^{-1} = \begin{pmatrix} \lambda_1 & 1 & 0 & & & & & 0 \\ & \ddots & \ddots & & & & & \\ & & \lambda_1 & & & & & \\ & & & \lambda_2 & 1 & & & \\ & & & & \ddots & 1 & & \\ & & & & & \lambda_2 & & \\ & & & & & & \lambda_s & 0 \quad 1 \\ 0 & & & & & & & \ddots \\ & & & & & & & \lambda_s \end{pmatrix}$$

then

$$BSB^{-1} = \begin{pmatrix} \lambda_1 & & 0 & & & 0 \\ & \ddots & & & & \\ & & \lambda_1 & & & \\ 0 & & & \lambda_2 & & \\ & & & & \ddots & \\ & & & & & \lambda_2 & & 0 \\ & & & & & & \lambda_s & \\ 0 & & & & & & & \ddots \\ & & & & & & & & \lambda_s \end{pmatrix}$$

and

$$BNB^{-1} = \begin{pmatrix} 0 & 1 & 0 & & & & 0 \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & \ddots & 1 & & \\ & & & & 0 & & \\ & & & & & 0 & 1 \\ 0 & & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}$$

This is useful for intuition and makes the statement believable (in the absence of a proof)

We also point out that $\pi_{c/u/s/h} \circ A = A \circ \pi_{c/u/s/h}$ by definition, and so likewise, as $e^{tA} = Id + tA + \frac{t^2A^2}{2!} + \dots$, we have $\pi_{c/u/s/h} \circ e^{tA} = e^{tA}\pi_{c/u/s/h}$. Thus in Prop 1, we will see $e^{At}\pi_c$ as a linear map from U_c to U_c, and likewise for $e^{At}\pi_s$ and $e^{At}\pi_u$ from $U_s \to U_s$, $U_u \to U_u$ resp.

as all norms on $\mathbb{R}^n$ are equivalent, we get the statement for $e^{At}\pi_{c/u/s} : \mathbb{R}^n \to \mathbb{R}^n$ by setting $\|x\| := \|x_c\|_c + \|x_s\|_s + \|x_u\|_u$ when $x = x_c + x_s + x_u$, with $x_c \in U_c$, $x_s \in U_s$, $x_u \in U_u$ and with our favourite norms $\|\cdot\|_c$, $\|\cdot\|_s$ and $\|\cdot\|_u$ on U_c, U_s and U_u respectively

As we will see in the proof of Prop. 1, the ε's come from the nilpotent parts. e.g. let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

So the ODE $\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ looks like $\dot{x} = y$, $\dot{y} = z$, $\dot{z} = 0$

(22) So $z(t) = z_0$, $y(t) = z_0 t + y_0$, $x(t) = \frac{z_0}{2}t^2 + y_0 t + x_0$

This can have ~~exponential~~ growth, e.g. ~~as~~ if $x_0, y_0, z_0 > 0$.
But it is polynomial, not exponential.

<u>Proof of Prop 1</u> We fix $\varepsilon > 0$ and write $A_c := \pi_c A \pi_c : U_c \circlearrowleft$ then we may see $e^{tA}\pi_c$ as a map from U_c to U_c which is given by e^{tA_c}. Write $A_c = S_c + N_c$ in Jordan Chevalley Decomposition. As $S_c \circ N_c = N_c \circ S_c$, we have

$$e^{tA_c} = e^{tN_c} \cdot e^{tS_c}, \text{ so } \|e^{tA_c}\| \le \|e^{tN_c}\| \cdot \|e^{tS_c}\|.$$

As S_c is semisimple with all eigenvalues on $i\mathbb{R}$, so

$$S_c \sim \underbrace{\begin{pmatrix} 0 & & & & 0 \\ & \ddots & & & \\ 0 & & \boxed{\begin{matrix} 0 & -\omega_1 \\ \omega_1 & 0 \end{matrix}} & & 0 \\ & & & \ddots & \\ 0 & & & & \boxed{\begin{matrix} 0 & -\omega_s \\ \omega_s & 0 \end{matrix}} \end{pmatrix}}_{D}$$

we have $\|e^{tA_c}\| \le C$ for some $C > 0$. More precise, if D is diagonal (or 2×2 block diagonal) then $\|D\| \le 1$ and we may write $A_c = BDB^{-1}$, so $e^{tA_c} = e^{tBDB^{-1}} = Be^{tD}B^{-1}$, so

$$\|e^{tA_c}\| \le \|B\|\|B^{-1}\|\,\|e^{tD}\| \le \|B\|\cdot\|B^{-1}\| =: C$$

Now we look at e^{tN_c}. Assume $N_c^{k+1} = 0$ for $k \ge 0$. Then

$$e^{tN_c} = Id + tN_c + \frac{t^2}{2!}N_c^2 + \dots + \frac{t^k}{k!}N_c^k \quad \text{So}$$

$$\begin{aligned}
\|e^{tN_c}\| &\le \|Id\| + |t|\|N_c\| + \frac{|t|^2}{2!}\|N_c\|^2 + \dots + \frac{|t|^k}{k!}\|N_c\|^k \\
&= 1 + |t|\|N_c\| + \dots + \frac{|t|^k}{k!}\|N_c\|^k \\
&= 1 + |\varepsilon t|\left(\frac{\|N_c\|}{\varepsilon}\right) + \frac{|\varepsilon t|^2}{2!}\frac{\|N_c\|^2}{\varepsilon^2} + \dots + \frac{|\varepsilon t|^k}{k!}\frac{\|N_c\|^k}{\varepsilon^k} \\
&\le \tilde{C}(\varepsilon)\left(1 + |\varepsilon t| + \frac{|\varepsilon t|^2}{2!} + \dots + \frac{|\varepsilon t|^k}{k!}\right)
\end{aligned}$$

where $\tilde{C}(\varepsilon) = \max\left(1, \frac{\|N_c\|}{\varepsilon}, \frac{\|N_c\|^2}{\varepsilon^2}, \dots, \frac{\|N_c\|^k}{\varepsilon^k}\right) > 0$

$$\begin{aligned}
\text{So } \|e^{tN_c}\| &\le \tilde{C}(\varepsilon)\left(1 + |\varepsilon t| + \dots + \frac{|\varepsilon t|^k}{k!}\right) \\
&\le \tilde{C}(\varepsilon)\left(1 + |\varepsilon t| + \frac{|\varepsilon t|^2}{2!} + \dots + \frac{|\varepsilon t|^k}{k!} + \frac{|\varepsilon t|^{k+1}}{(k+1)!} + \dots\dots\right) \\
&= \tilde{C}(\varepsilon)\, e^{|\varepsilon t|} = \tilde{C}(\varepsilon)\, e^{\varepsilon|t|}
\end{aligned}$$

Thus putting things together:

$$\begin{aligned}
\|e^{tA_c}\| &\le \|e^{tN_c}\| \cdot \|e^{tS_c}\| \\
&\le C \cdot \tilde{C}(\varepsilon)\, e^{\varepsilon|t|} \\
&= M(\varepsilon)\, e^{\varepsilon|t|} \quad \text{if } M(\varepsilon) := C \cdot \tilde{C}(\varepsilon)
\end{aligned}$$

(23) The cases $\|e^{At}\pi_s\|$ and $\|e^{At}\pi_u\|$ go similarly: Denote by A_s the restriction of A to U_s, and let $A_s = N_s + S_s$ be its Jordan Chevalley decomposition. Then there are invertible $B : U_s \circlearrowleft$ and a matrix

$$D = \begin{pmatrix} \lambda_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \lambda_1 & & & & \\ & & & \lambda_2 & & & \\ & & & & \ddots & & \\ & & & & & \lambda_2 & \\ 0 & & & & & & \begin{pmatrix} \lambda_s & \omega_s \\ -\omega_s & \lambda_s \end{pmatrix} \\ & & & & & & & \ddots \end{pmatrix}$$

with all $\lambda_i < 0$ and $\beta_- = \max\{\lambda_i\}$

such that $S_s = B D B^{-1}$

and so $e^{tS_s} = B e^{tD} B^{-1}$

From this $\|e^{tS_s}\| \leq \|B\|\|B^{-1}\|\,\|e^{tD}\| = \|B\|\|B^{-1}\|\, e^{t\beta_-}$ for $t \geq 0$

also as before $\|e^{tN_s}\| \leq \tilde{C}(\varepsilon)\, e^{\varepsilon|t|}$ for all t, so $\|e^{tN_s}\| \leq \tilde{C}(\varepsilon)\, e^{\varepsilon t}$

and thus $\|e^{tA_s}\| \leq \|e^{tN_s}\|\|e^{tS_s}\| \leq M(\varepsilon)\, e^{t(\beta_- + \varepsilon)}$ for $M(\varepsilon) = \|B\|\|B^{-1}\|\,\tilde{C}(\varepsilon)$.

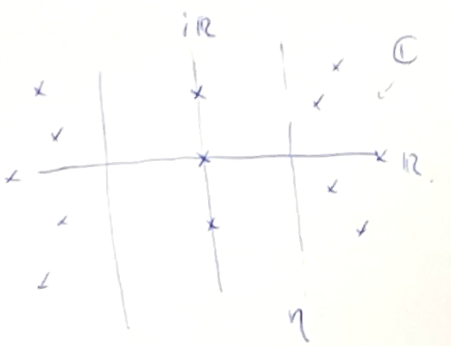
The case $\|e^{tA}\pi_u\|$ is completely analogous, or deduce from the stable case by sending $t \mapsto -t$, $A \mapsto -A$. This completes the proof □

Prop 1 will be very important later on. For now, we explore a consequence for linear systems:

Lem 2 Let $A \in \mathrm{Mat}(n;\mathbb{R})$ and denote by $\varphi_t(x)$ (so $\varphi_t(x) = e^{At}x$) the flow of the ODE $\dot{x} = Ax$. Let U_c, U_h be as defined before using A, and denote by $\pi_h : \mathbb{R}^n \to U_h$ the projection onto U_h along U_c. Then we have

$$U_c = \left\{ x \in \mathbb{R}^n \;\middle|\; \sup_{t \in \mathbb{R}} \|\pi_h \varphi_t(x)\| < \infty \right\} = \left\{ x \in \mathbb{R}^n \;\middle|\; \sup_{t \in \mathbb{R}} e^{-\eta|t|} \|\varphi_t(x)\| < \infty \right\}$$

where η is any real number satisfying $0 < \eta < \beta$ with $\beta = \min\{\beta_+, -\beta_-\}$



(So all eigenvalues λ of A are either on $i\mathbb{R}$, or satisfy $|\mathrm{Re}(\lambda)| > \eta$)

(24) Proof of Lemma 2 • If $x \in U_c$ then $\varphi_t(x) = e^{tA}x \in U_c$ for all $t \in \mathbb{R}$,

so $\sup_{t \in \mathbb{R}} \|\pi_h \varphi_t(x)\| = \sup_{t \in \mathbb{R}} \|0\| = 0 < \infty$.

• Now suppose $x \in \mathbb{R}^n$ is such that $\sup_{t \in \mathbb{R}} \|\pi_h \varphi_t(x)\| < \infty$, say $\sup_{t \in \mathbb{R}} \|\pi_h \varphi_t(x)\| = C$.

then we may write $x = x_c + x_h$ for $x_h = \pi_h(x) \in U_h$, $x_c \in U_c$.

We get $\|\varphi_t(x)\| = \|e^{tA}(x_c + x_h)\| = \|e^{tA}x_c + e^{tA}x_h\| \leq \|e^{tA}x_c\| + \|e^{tA}x_h\|$

$= \|e^{tA}x_c\| + \|e^{tA}\pi_h x\| = \|e^{tA}x_c\| + \|\pi_h e^{tA}x\| = \|e^{tA}x_c\| + \|\pi_h \varphi_t(x)\|$

$\leq M(\eta)e^{\eta|t|} + C$ by Prop 1.

$\leq M(\eta)e^{\eta|t|} + Ce^{\eta|t|}$ (as $e^{\eta|t|} \geq 1$ for all $t \in \mathbb{R}$)

So $e^{-\eta|t|}\|\varphi_t(x)\| \leq M(\eta) + C$ thus $\sup_{t \in \mathbb{R}} e^{-\eta|t|}\|\varphi_t(x)\| < \infty$

• Now suppose $x \in \mathbb{R}^n$ satisfies $\sup_{t \in \mathbb{R}} e^{-\eta|t|}\|\varphi_t(x)\| < \infty$, so

$\|\varphi_t(x)\| \leq Ce^{\eta|t|}$ for some $C > 0$. Then for $t \leq 0$: (by Prop 2)

$$\|\pi_u x\| = \|e^{At}\pi_u e^{-At}x\| \leq M(\varepsilon)e^{(\beta_+ - \varepsilon)t}\|e^{-At}x\|$$
$$= M(\varepsilon)e^{(\beta_+ - \varepsilon)t}\|\varphi_{-t}(x)\|$$
$$\leq M(\varepsilon)e^{(\beta_+ - \varepsilon)t} \cdot C \cdot e^{\eta|-t|}$$

for any $\varepsilon > 0$. $\longrightarrow$

$$= M(\varepsilon) \cdot C \cdot e^{(\beta_+ - \varepsilon)t}e^{-\eta t} \quad (\text{as } t \leq 0, \text{ so } |-t| = -t)$$
$$= M(\varepsilon) \cdot C \; e^{(\beta_+ - \eta - \varepsilon)t}$$

So choose ε such that

$\varepsilon < \beta - \eta \leq \beta_+ - \eta$ So $(\beta_+ - \eta - \varepsilon) > 0$

Thus $\|\pi_u x\| \leq M(\varepsilon) \cdot C \cdot e^{(\beta_+ - \eta - \varepsilon)t}$ for all $t \leq 0$. So let $t \to -\infty$

and we get $\|\pi_u x\| \leq 0 \implies \underline{\pi_u x = 0}$.

Similarly, for $t \geq 0$ and $\varepsilon < -\beta_- - \eta$, so $\beta_- + \eta + \varepsilon < 0$:

$$\|\pi_s x\| \leq M(\varepsilon)e^{(\beta_- + \varepsilon)t} \cdot Ce^{\eta t} = M(\varepsilon)\,C\,e^{(\beta_- + \eta + \varepsilon)t}$$

as $t \to \infty$: $\|\pi_s x\| \leq 0 \implies \underline{\pi_s x = 0}$

β_- η

So $x = x_c + \pi_u x + \pi_s x = x_c + 0 + 0 = x_c \in U_c$

This completes the proof | Note that Lem 2 confirms our previous statement that all bounded solutions lie on U_c, as $\varphi_t(x)$ bounded implies $\pi_h \varphi_t(x)$ bounded $(\|\pi_h \varphi_t(x)\| \leq \|\pi_h\|\|\varphi_t(x)\|$ and so $x \in U_c$

$-\eta > \beta_-$

so $\beta_- + \eta < 0$

$\beta_- + \eta + \varepsilon < 0$

(25) We may now finally state our main result. It says that for systems "sufficiently close to linear" (but still with nonlinear terms) we have an invariant manifold closely mirroring the space U_c and its properties as laid out in Lemma 2:

Thm 3 [Center Manifold Reduction] Given $A \in Mat(n, \mathbb{R})$, denote by W_c and W_h its center and hyperbolic subspaces (named U_c & U_h before). Let $k \geq 1$ be given. There exists $\varepsilon = \varepsilon(A, k) > 0$ such that if a non-linearity $G: \mathbb{R}^n \to \mathbb{R}^n$ satisfies $G(0) = 0$, $DG(0) = 0$ and

$$1) \quad \sup_{x \in \mathbb{R}^n} \|D^j G(x)\| < \infty \quad \text{for } 0 \leq j \leq k$$

$$2) \quad \sup_{x \in \mathbb{R}^n} \|DG(x)\| < \varepsilon$$

Then the system $\dot{x} = Ax + G(x)$ has the following property: There exists a function $\Psi: W_c \to W_h$ of class C^k s.t. its ~~flow~~ graph $\{x_c + \Psi(x_c) \mid x_c \in W_c\}$ is flow-invariant. Ψ moreover is bounded and satisfies $\Psi(0) = 0$ and $D\Psi(0) = 0$. We usually refer to this flow invariant set as the Center Manifold of the system:

• $$M_c = \{x_c + \Psi(x_c) \mid x_c \in W_c\}$$

We moreover have

•• $$M_c = \{x \in \mathbb{R}^n : \sup_{t \in \mathbb{R}} \|\pi_h \varphi_t(x)\| < \infty\}$$

where $\varphi_t(x)$ denotes the flow of the system $\dot{x} = Ax + G(x)$ and $\pi_h: \mathbb{R}^n \to W_h$ is the projection w.r.t $\mathbb{R}^n = W_c \oplus W_h$. Also:

••• $$M_c = \{x \in \mathbb{R}^n : \sup_{t \in \mathbb{R}} e^{-\eta|t|} \|\varphi_t(x)\| < \infty\}$$ with $\eta \in (0, \beta)$ (with β defined by A as before)

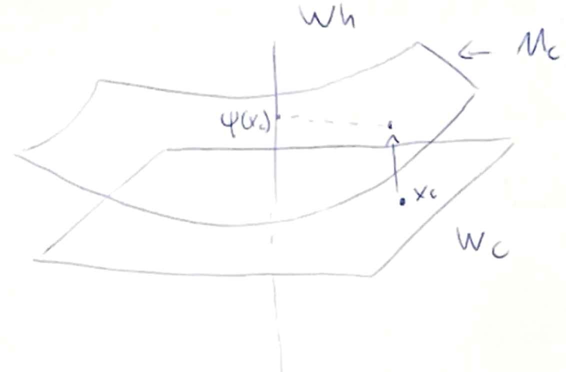
Finally: if $\phi: W_c \to W_h$ is continuous, bounded and such that the set $\{x_c + \phi(x_c) \mid x_c \in W_c\}$ is flow-invariant, then $\phi = \Psi$

Note that, if $G \equiv 0$ (so if the ODE is $\dot{x} = Ax$, so linear) then 1) & 2) are satisfied and $\Psi \equiv 0$, so $M_c = W_c$

by uniqueness

and we retrieve Lemma 2

(26) In general, the picture is:



Note that the set: $\{x \in \mathbb{R}^n \mid \sup_{t \in \mathbb{R}} \|\pi_h \varphi_t(x)\| < \infty\}$ is flow-invariant by construction. For, if $\|\pi_h \varphi_t(x)\| < C$ for some $C > 0$ and all $t \in \mathbb{R}$, then given $t_0 \in \mathbb{R}$, $\|\pi_h \varphi_t(\varphi_{t_0}(x))\| = \|\pi_h \varphi_{t+t_0}(x)\| < C$ for all $t \in \mathbb{R}$.

(Note as well that, since G is assumed bounded, solutions to $\dot{x} = Ax + G(x)$ exist for all $t \in \mathbb{R}$!)

If $\varphi_t(x)$ is a bounded solution, say $\|\varphi_t(x)\| < D$ for some $D > 0$ and all $t \in \mathbb{R}$, then clearly $\|\pi_h \varphi_t(x)\| \leq \|\pi_h\| \, \|\varphi_t(x)\| \leq \|\pi_h\| D < \infty$ so M_c contains all bounded solutions. However, M_c can (and will often) contain unbounded solutions too. Consider for instance the linear system $\dot{x} = Ax$ (so $G \equiv 0$, so $M_c = W_c$) where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and so $M_c = \mathbb{R}^2$. We already saw that $\dot{x} = Ax$ $\left(\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right)$ has unbounded solutions $y(t) = y_0 \neq 0$, $x(t) = y_0 t + x_0$

Some more remarks on the theorem: the system $\dot{x} = F(x) := Ax + G(x)$ has a steady state point in 0: $F(0) = G(0) = 0$, so $0 \in M_c$, and so $\psi(0) = 0$ necessarily. Thus that follows

Regarding the uniqueness: if $\phi : W_c \to W_h$ is continuous and bounded, then say $\|\phi(x_c)\| < C$ for some $C > 0$ and all $x_c \in W_c$. If in addition $\{x_c + \phi(x_c) \mid x_c \in W_c\}$ is flow invariant, then necessarily for $x_0 = x_c + \phi(x_c)$, we have $\pi_h(\varphi_t(x_0)) = \phi(\pi_c(\varphi_t(x_0)))$.

so $\|\pi_h(\varphi_t(x_0))\| < C$.

So $\|\pi_h \varphi_t(x_0)\|$ is bounded

and thus $x_0 = x_c + \phi(x_c) \in \{x_c + \psi(x_c) \mid x_c \in W_c\}$

But then $\underbrace{\pi_h(x_0)}_{= \phi(x_c)} = \underbrace{\psi(\pi_c(x_0))}_{= \psi(x_c)}$ for all x_c, so $\phi \equiv \psi$.

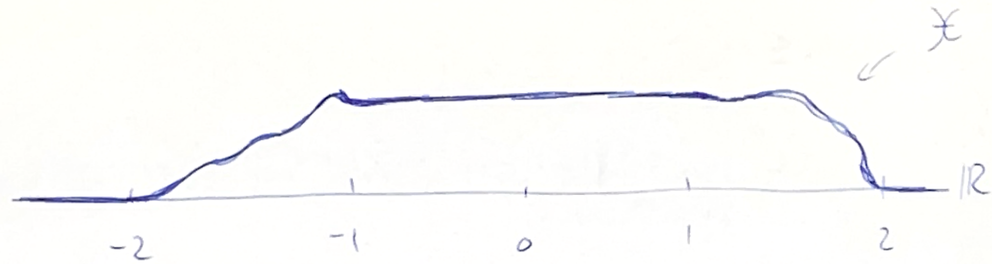
27 Note that the conditions on G in Thm 3 seem a bit harsh. When are non-linearities ever bounded?

The way we will still be able to use Thm 3 in bifurcation analysis is as follows: As we are often only interested in the behavior of $\dot{x} = Ax + G(x) = F(x)$ near the fixed point 0, we may replace $F(x)$ by another vectorfield $\tilde{F}(x)$ that agrees with F in some open neighborhood of $0 \in \mathbb{R}^n$.

To this end, let $\chi : \mathbb{R}^n \to \mathbb{R}$ be some C^∞ bump function

That is, assume $\chi(x) = 1$ for all $x \in B_1(0) = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$
and $\chi(x) = 0$ for all $x \notin B_2(0) = \{ \cdots \|x\| < 2\}$

So for $n = 1$ it may look like



Then we have:

$\leftarrow k \geq 1$

Lem 4. Given $G : \mathbb{R}^n \to \mathbb{R}^n$ of class C^k, the map $\tilde{G}_\rho : \mathbb{R}^n \to \mathbb{R}^n$, $\tilde{G}_\rho(x) = \chi(\rho^{-1} x) G(x)$ agrees with G on the open set $B_\rho(0)$ (where $\rho \in \mathbb{R}_{>0}$)

Suppose $G(0) = 0$ & $DG(0) = 0$, then given $\varepsilon > 0$, there is a $\tilde{\rho}_\varepsilon > 0$ such that $\tilde{G}_\rho$ satisfies the two conditions 1) and 2) in Thm 3, so

1) $\sup_{x \in \mathbb{R}^n} \|D^j \tilde{G}_\rho(x)\| < \infty$ for $0 \leq j \leq k$

2) $\sup_{x \in \mathbb{R}^n} \|D\tilde{G}_\rho(x)\| < \varepsilon$

If $\rho < \tilde{\rho}_\varepsilon$.

Thus Center Manifold Reduction may be applied to the system $\dot{x} = Ax + \tilde{G}_\rho(x)$, which agrees with $\dot{x} = Ax + G(x)$ locally around 0.

Proof of Lemma 4. it is clear that $\chi(p^{-1}x)G(x)=G(x)$ when $\chi(p^{-1}x)=1$, which is (at least) when $\|p^{-1}x\|<1$, so when $\|x\|<p$, so for $x\in B_p(0)$. also, as $\tilde G_p(x)=0$ when $x\notin B_{2p}(0)$, we immediately see that

1) $\sup_{x\in\mathbb{R}^n}\|D^j\tilde G_p(x)\|<\infty$ for $0\le j\le k$, for all $p>0$.

Now note that $D\tilde G_p(x)=\chi(p^{-1}x)DG(x)+p^{-1}D\chi(p^{-1}x)G(x)$ as $G(0)=0$ and by the Mean value theorem

$$\|G(x)\|=\|G(x)-G(0)\|\le\sup_{s\in[0,1]}\|DG(sx)x\|\le\|x\|\sup_{s\in[0,1]}\|DG(sx)\|$$

$$
\begin{aligned}
\text{So } \sup_{x\in\mathbb{R}^n}\|D\tilde G_p(x)\| &\le \sup_{\|x\|\le 2p}\|D\tilde G_p(x)\| \\
&\le \sup_{\|x\|\le 2p}\left(|\chi(p^{-1}x)|\,\|DG(x)\|+p^{-1}\|D\chi(p^{-1}x)\|\,\|G(x)\|\right) \\
&\le \sup_{\|x\|\le 2p}\left(C_1\|DG(x)\|+p^{-1}C_2\|G(x)\|\right) \\
&\le C_1\sup_{\|x\|\le 2p}\|DG(x)\|+C_2p^{-1}\sup_{\|x\|\le 2p}\|x\|\sup_{s\in[0,1]}\|DG(sx)\| \\
&\le C_1\sup_{\|x\|\le 2p}\|DG(x)\|+C_2\,p^{-1}\cdot 2p\cdot\sup_{\|x\|\le 2p}\|DG(x)\| \\
&= (C_1+2C_2)\sup_{\|x\|\le 2p}\|DG(x)\|
\end{aligned}
$$

where

$C_1:=\sup_{x\in\mathbb{R}^n}\chi(x)$

$C_2:=\sup_{x\in\mathbb{R}^n}\|D\chi(x)\|$

as $DG(0)=0$ and G is C^1 (so DG is continuous), we find

$$\lim_{p\to 0}\|D\tilde G_p(x)\|=\lim_{p\to 0}(C_1+2C_2)\sup_{\|x\|\le 2p}\|DG(x)\|=0.$$

So given ε, for p small enough, $\|D\tilde G_p(x)\|<\varepsilon$.
This completes the proof.

Before we show how to use Thm 3 in an example, we now look at the coefficients of φ in more detail. Note, by the way, that the construction surrounding Lemma 4 depends on all kinds of choices, and that

Now Suppose we have a C^k map $\Psi: W_c \to W_h$, with W_c & W_h determined by $A \in Mat(\mathbb{R}, n)$. Suppose $\Psi(0)=0$ and that $\{x_c+\Psi(x_c) \mid x_c \in W_c\}$ is invariant under the flow of $\dot{x} = f(x) = Ax + g(x)$, $(g(0)=0,\ Dg(0)=0)$ on $\mathbb{R}^n = W_c \oplus W_h$. Let's see what this means for the Taylor-coefficients of Ψ. As usual, denote by π_c & π_h the projections onto W_c & W_h, respectively, along $\mathbb{R}^n = W_c \oplus W_h$. We may write $\dot{x} = f(x)$ as

$$\dot{x}_c = A_c x_c + \pi_c g(x_c, x_h)$$
$$\dot{x}_h = A_h x_h + \pi_h g(x_c, x_h)$$

where $x_{c/h} = \pi_{c/h}(x)$ and $A = \begin{pmatrix} A_c & 0 \\ 0 & A_h \end{pmatrix}$ w.r.t. $\mathbb{R}^n = W_c \oplus W_h$.

What does a vector tangent to $M_c = \{x_c + \Psi(x_c) \mid x_c \in W_c\}$ look like? well, we may describe them by $\frac{d}{dt}\big|_{t=0} (x_c + tv_c) + \Psi(x_c + tv_c)$

$$= v_c + D\Psi(x_c)v_c, \quad v_c \in W_c.$$

Thus for M_c to be flow-invariant, we need

$$\pi_h f(x_c + \Psi(x_c)) = D\Psi(x_c)\pi_c f(x_c + \Psi(x_c)) \quad \text{So}$$

$$A_h x_h + \pi_h g(x_c + \Psi(x_c)) = D\Psi(x_c)\big(A_c x_c + \pi_c g(x_c + \Psi(x_c))\big)$$

$$\Rightarrow A_h \Psi(x_c) + \pi_h g(x_c + \Psi(x_c)) = D\Psi(x_c)\big(A_c x_c + \pi_c g(x_c + \Psi(x_c))\big)$$

Write $\Psi(x_c) = Bx_c + O(|x_c|^2)$ with B linear, then comparing linear terms alone:

$$A_h B x_c = B A_c x_c + O(|x_c|^2) \Rightarrow A_h B = B\cancel{A}A_c \text{ on } W_c.$$

Or

$$\begin{tikzcd} W_c \arrow[r, "A_c"] \arrow[d, "B"'] \arrow[dr, phantom, "\circlearrowright"] & W_c \arrow[d, "B"] \\ W_h \arrow[r, "A_h"] & W_h \end{tikzcd}$$

as diagrams.

We claim that necessarily $B = 0$:

<u>Lemma</u>: Suppose $A_1: \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$, $A_2: \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$, $B: \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ are linear such that A_1 & A_2 have no eigenvalues in common and suppose $A_2 B = B A_1$, then $B = 0$.

<u>Proof</u> Expand A_1, A_2, B to $\mathbb{C}$-linear maps on $\mathbb{C}^{n_1}, \mathbb{C}^{n_2}$. Denote by $\mathcal{L}_{\mathbb{C}}(n_1, n_2)$ the space of $\mathbb{C}$-linear maps from $\mathbb{C}^{n_1}$ to $\mathbb{C}^{n_2}$. (so $B \in \mathcal{L}_{\mathbb{C}}(n_1, n_2)$)

(30) Consider the $\mathbb{C}$-linear map $L : \mathcal{L}_{\mathbb{C}}(n_1, n_2) \circlearrowleft$ given by $LX = A_2 X - X A_1$, so $B \in \ker(L)$

Claim 1. If $v_1 \dots v_{n_1}$ is a basis for $\mathbb{C}^{n_1}$ (over $\mathbb{C}$), and $w_1 \dots w_{n_2}$ is a basis for $\mathbb{C}^{n_2}$ (over $\mathbb{C}$) then a basis for $\mathcal{L}_{\mathbb{C}}(n_1, n_2)$ is given by

$$\mathcal{L}_{\mathbb{C}}(n_1,n_2) \cong \mathbb{C}^{n_2} \otimes \mathbb{C}^{*n_1} \qquad \{ w_i v_j^* \mid 0 < i \le n_2,\ 0 < j \le n_1 \}$$

where $v_1^* \dots v_{n_1}^*$ is the dual basis to $v_1 \dots v_{n_1}$

i.e. $v_i^*(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} = \delta_{ij}$

Proof. Note that $\dim \mathcal{L}_{\mathbb{C}}(n_1, n_2) = n_1 n_2$. if

$$\sum_{j=1}^{n_1} \sum_{i=1}^{n_2} a_{ij} w_i v_j^* = 0$$

for some $a_{ij} \in \mathbb{C}$, then given $0 < k \le n_1$

$$0 = \sum_{j=1}^{n_1} \sum_{i=1}^{n_2} a_{ij} w_i v_j^*(v_k) = \sum_{j=1}^{n_1} \sum_{i=1}^{n_2} a_{ij} w_i \delta_{jk} = \sum_{i=1}^{n_2} a_{ik} w_i$$

So (as the w_i are a basis) $a_{1k} = a_{2k} = \dots a_{n_2 k} = 0$.
This holds for all k, so the $w_i v_j^*$ are linearly independent.
counting dimensions, we see that we are good.

Now suppose $v_1 \dots v_{n_1}$ is a basis in which A_1 is upper-triangular, and $w_1 \dots w_{n_2}$ is a basis -- A_2 is ~~upper~~ lower-triangular. i.e.

$$A_1 v_i = d_i v_i + \text{" terms in } v_{i-1}, v_{i-2} \dots v_1 \text{"}$$
$$A_2 w_j = \tilde{d}_j w_j + \text{" terms in } w_{j+1} \dots w_{n_2} \text{"}.$$

Note that the d_i are the eigenvalues of A_1 and that $\tilde{d}_j$ the eigenvalues of A_2.

(So $d_i \neq \tilde{d}_j$ for all i, j, by assumption)

Order the basis $w_i v_j^*$ such that $w_i v_j^* < w_l v_k^*$ if $i + j < l + k$.

e.g. $w_1 v_1^* < w_1 v_2^* < w_2 v_1^* < w_2 v_2^* < w_1 v_3^* < w_3 v_1^* < \dots$

Claim: L is ~~upper~~ lower triangular w.r.t this orderded basis. the diagonal entries are $\tilde{d}_j - d_i \neq 0$.

(31) Proof: Recall $A_1 v_i = d_i v_i$ + "terms lower in i"

So $v_i^*(A_1 v_i) = v_i^*(d_i v_i +$ "terms lower in i"$) = d_i = (v_i^* \circ A_1)(v_i)$

and $v_i^*(A_1 v_k) = v_i^*(d_k v_k +$ "terms lower"$) = 0 = (v_i^* \circ A_1)(v_k)$

if $k < i$.

So $v_i^* \circ A_1 = d_i v_i^*$ + "terms in $v_{i+1}^*, \dots v_{n_1}^*$"

Thus $\mathcal{L}\, w_j v_i^* = A_2 w_j v_i^* - w_j (v_i^* \circ A_1)$

$= (\tilde{d}_j w_j +$ "terms in $w_{j+1} \dots w_{n_2}$"$) v_i^*$

$- w_j (d_i v_i^* +$ "terms in $v_{i+1}^* \dots v_{n_1}^*$"$)$

$= (\tilde{d}_j - d_i) w_j v_i^*$ + "higher order terms"

Since $\tilde{d}_j - d_i \neq 0$ for all $1 \leq i \leq n_1$, $1 \leq j \leq n_2$. and since $\mathcal{L}$ is lower-triangular, we see that $\mathcal{L} : \mathcal{L}_{\mathbb{C}}(n_1, n_2)$ is injective, thus bijective. But, $\mathcal{L}(B) = 0$ so $B = 0$ $\square$.

Note that we only used that A_1 & A_2 (so A_c & A_h) have no eigenvalues in common!

Back to $\Psi : W_c \to W_h$, apparently

$$\Psi(x_c) = \underbrace{\Psi^{(2)}(x_c)}_{\text{2nd order}} + \underbrace{\Psi^{(3)}(x_c)}_{\text{3rd order}} + \dots$$ (No constant or linear terms around 0; $\Psi(0) = 0$, $D\Psi(0) = 0$)

Back to our original equation

$$A_h \Psi(x_c) + \pi_h g(x_c, \Psi(x_c)) = D\Psi(x_c)\left(A_c x_c + \pi_c g(x_c + \Psi(x_c))\right)$$

Write $\pi_h g(x_c, x_h) = g^{(2,0)}(x_c) + g^{(1,1)}(x_c, x_h) + g^{(0,2)}(x_h) + \mathcal{O}(|x_c, x_h|^3)$

Then $\pi_h g(x_c, \Psi(x_c)) = g^{(2,0)}(x_c) + \mathcal{O}(|x_c|^3)$

and $\left.\begin{array}{l} \pi_c g(x_c, \Psi(x_c)) = \mathcal{O}(|x_c|^2), \\ D\Psi(x_c) = \mathcal{O}(|x_c|) \end{array}\right\}$ $D\Psi(x_c)(\pi_c g(x_c + \Psi(x_c))) = \mathcal{O}(|x_c|^3)$.

Thus for 2nd order:

$$A_h \Psi^{(2)}(x_c) + g^{(2,0)}(x_c) = D\Psi^{(2)}(x_c) A_c x_c + \mathcal{O}(|x_c|^3)$$

So to solve:

$$A_h \Psi^{(2)}(x_c) - D\Psi^{(2)}(x_c) A_c x_c = -g^{(2,0)}(x_c).$$

It turns out this has a unique solution $\Psi^{(2)}(x_c)$.

More generally: If $\Psi^{(2)}(x_c) \dots \Psi^{(l)}(x_c)$ are known, (as are $\pi_{c/h}\, g(x_c, x_h)$) then the order $(l+1)$ terms of $\pi_h\, g(x_c, \Psi(x_c))$ are known, as are those of $D\Psi(x_c)\big(\pi_c\, g(x_c, \Psi(x_c))\big)$

So we get the equation

☆ $$A_h \Psi^{(l+1)}(x_c) - D\Psi^{(l+1)}(x_c) A_c x_c = \text{"something known"}$$

Again this has a unique solution $\Psi^{(l+1)}(x_c)$.

Thus. The Taylor-expansion of Ψ at 0 can be solved for (and is therefore unique!) up to any order that the regularity of Ψ permits! (i.e. if Ψ is C^k but not C^{k+1} then the $k+1$ order term makes little sense)

- For the uniqueness claim on eq. ☆, see e.g.
  [H.K. Wimmer, The equation $(g(x))_x\, ax - b\, g(x) = h(x)$, J. Math. Anal. Appl. 67 (1979), 198-204]

- Note that we only assumed a map $\Psi: W_c \to W_h$ exists (maybe only locally defined around 0) such that $\Psi(0) = 0$ and the graph $M_c := \{x_c + \Psi(x_c) \mid x_c \in \mathring{U} \subseteq W_c\}$ is invariant under the flow of $\dot{x} = f(x) = Ax + g(x)$.

(We concluded that $D\Psi(0) = 0$ and that all higher-order Taylor coefficients at 0, if they exist, are fixed by g)

We call a graph M_c of such a Ψ a <u>Center-Manifold</u> of the system $\dot{x} = f(x)$.

As the Taylor-coefficients around 0 are fixed, it is tempting to think center-manifolds themselves are (locally) unique. However, this is false, as the following example shows:

Here the center manifolds will even be globally defined and C^∞.

(33) Consider the ODE on $\mathbb{R}^2$, given by

$$\dot{x} = -x^3$$
$$\dot{y} = -y$$

around $(0,0)$, $W_c = \{(x,y) \mid y=0\} = \text{"}\{(x,0)\}\text{"}$

$W_h = \{(x,y) \mid x=0\} = \{(0,y)\}$

Fix any $\alpha, \beta \in \mathbb{R}$, and define

$$\psi_{\alpha,\beta}(x) = \begin{cases} \alpha e^{-\frac{1}{2x^2}} & \text{for } x<0 \\ 0 & \text{for } x=0 \\ \beta e^{-\frac{1}{2x^2}} & \text{for } x>0 \end{cases} \quad (9)$$

We claim that $M_c^{\alpha,\beta} := \{(x, \psi_{\alpha,\beta}(x)) \mid x \in \mathbb{R}\}$ is invariant for the given ODE. To see why, note that it suffices to check that the flow is tangential to (9). This is clear for $(0,0)$, on $(x, \psi_{\alpha,\beta}(x))$ with $x>0$: $T_{(x,\psi_{\alpha,\beta}(x))}M_c^{\alpha,\beta} = \{(v, D\psi_{\alpha,\beta}(x)v) \mid v \in \mathbb{R}\}$

$= \{(v, \frac{\beta}{x^3} e^{-\frac{1}{2x^2}} v) \mid v \in \mathbb{R}\}$ To check: $(-x^3, -\beta e^{-\frac{1}{2x^2}}) \in T_{(x,\psi_{\alpha,\beta})} M_c^{\alpha,\beta}$

But indeed if $v = -x^3$ then $\frac{\beta}{x^3} e^{-\frac{1}{2x^3}} v = \frac{\beta}{x^3} e^{-\frac{1}{2x^3}} \cdot (-x^3)$

$= -\beta e^{-\frac{1}{2x^2}}$ ✓

So every choice of $\alpha, \beta \in \mathbb{R}$ defines a C^∞ centermanifold. Note that these are all distinct in any neighborhood of $(0,0) \in \mathbb{R}^2$

Here another way of seeing what's going on:

Solving the ODE explicitly: $y(t) = y(0)e^{-t}$, (1)

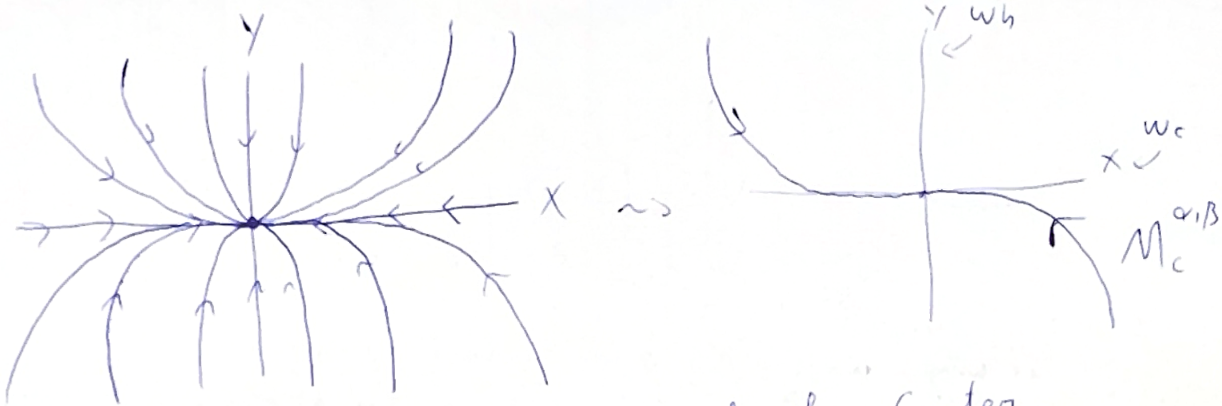
$$\frac{dx}{dt} = -x^3 \text{ "}\Rightarrow\text{" } dt = -\frac{1}{x^3}dx \text{ "}\Rightarrow\text{" } \int dt = \int -\frac{1}{x^3}dx$$

$$t = \frac{1}{2x^2} + c \quad (2)$$

(So $2x^2 = \frac{1}{t-c}$, $x = \pm\frac{1}{\sqrt{2(t-c)}} = \text{sgn}(x(0)) \frac{1}{\sqrt{2t + x(0)^2}}$ (3))

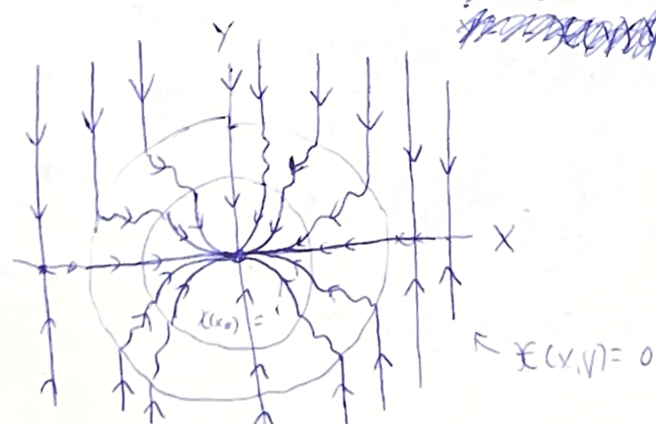
already from (1) & (2):

$$y(t) = y(0)e^{-t} = y(0)\, e^{-\frac{1}{2x^2}} \cdot e^{-c}$$
$$= \gamma e^{-\frac{1}{2x^2}} \text{ where } \gamma = y(0)e^{-c}$$



So we are just fully free to make the center-manifold out of any orbit for positive X, any orbit for negative X, and the orbit (0,0).
Which one would we have found locally, using our techniques? Well, if $\chi : \mathbb{R}^2 \to \mathbb{R}$ is a bump function then we would consider the system ~~[illegible]~~

$$\dot{x} = -\chi(x,y)x^3$$
$$\dot{y} = -y$$



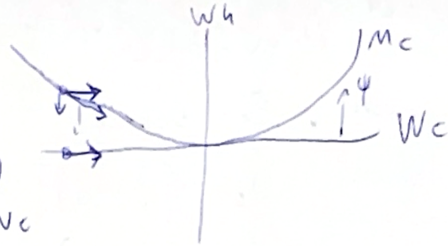
Note that for big enough $\|(x,y)\|$, the ODE becomes $\dot{x} = 0$, $\dot{y} = -y$
so to have unbounded Wh-behavior (y-behavior) (as well as an invariant graph over X) we need $\psi(x) = 0$ at least eventually. But of course $\psi(x) = 0$ everywhere works, so that is the (necesserally unique!) option we'd find: $M_c = W_c$

Allright, so we can put our finger on a (locally defined) invariant manifold containing all bounded solutions (locally). How does this help us, say in a bifurcation analysis?

Well, working on a manifold is best done using a chart, luckily we have a global one.

35 **Observation**: a chart for M_c is given by $W_c \ni X_c \longmapsto (X_c, \Psi(X_c)) \in M_c$

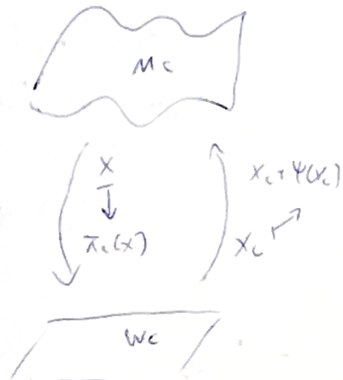
with inverse $M_c \ni \underset{X}{\underbrace{(X_c, X_h)}} \longmapsto X_c = \pi_c(X) \in W_c$



So we may pullback the vectorfield $F|_{M_c}$ to W_c

We get the vectorfield R on W_c given by

$$\boxed{R(X_c) = \pi_c F(X_c + \Psi(X_c))}$$



I will usually refer to R as the reduced vectorfield of the system $\dot{x} = F(X)$.

Note that $\dot{X}_c = R(X_c)$ is an ODE on W_c that is conjugate the the restriction of $\dot{x} = F(x)$ to M_c

Writing $F(X) = F(X_c + X_h) = \begin{pmatrix} A_c X_c + g_c(X_c + X_h) \\ A_h X_h + g_h(X_c + X_h) \end{pmatrix}$

We have $R(X_c) = A_c X_c + g_c(X_c + \Psi(X_c))$

Thus $R(0) = 0$, $DR(0) = A_c$, Moreover, if

$$g_c(X_c + X_h) = g_c^{(2,0)}(X_c) + g_c^{(1,1)}(X_c, X_h) + g_c^{(0,2)}(X_h) + \mathcal{O}(\|X_c + X_h\|^3)$$

then, since $\Psi(X_c) = \mathcal{O}(|X_c|^2)$, we get

$$R(X_c) = \underbrace{A_c X_c}_{\text{1st order}} + \underbrace{g_c^{(2,0)}(X_c)}_{\text{2nd order}} + \mathcal{O}(|X_c|^3)$$

Of course, higher order terms will depend on Ψ, in general.

Example: [from Scholarpedia.org] Consider the ODE on $\mathbb{R}^2$ given by

$$\left.\begin{aligned} \dot{x} &= a x^3 + x y - x y^2 \\ \dot{y} &= -y + b x^2 + x^2 y \end{aligned}\right\} = F_{a,b}(X, Y) \quad , (\text{so } F_{a,b}(0,0) = 0$$

for some $a, b \in \mathbb{R}$, assume $a + b \neq 0$ (this will be clear later). Determine those a, b for which $(0,0) \in \mathbb{R}^2$ is stable

Note that $DF_{a,b}(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ So the linearity gives us no information. So we will use Center Manifold Reduction

Note that $W_c = \{(x,0) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$

$W_h = \{(0,y) \mid y \in \mathbb{R}\} \subseteq \mathbb{R}^2$

We first need to calculate some Taylor expansion terms of ψ:

Let us write $\psi(x) = lx + cx^2 + dx^3 + O(|x|^4)$. As before, we want $\pi_h F_{a,b}(x_c + \psi(x_c)) = D\psi(x_c)\pi_c F_{a,b}(x_c + \psi(x_c))$

(note that we already know $l = 0$; this is just a sanity-check)

So $-\psi(x) + bx^2 + x^2\psi(x) = D\psi(x)\left(ax^3 + x\psi(x) - x\psi(x)^2\right)$

(where $x = x_c$)

Thus

$$-lx - cx^2 - dx^3 + bx^2 + x^2(lx) + O(|x|^4)$$

$$= (l + 2cx + 3dx^2)\left(ax^3 + lx^2 + cx^3 - x(l^2x^2)\right) + O(|x|^4)$$

1st order: $-lx = 0 \Rightarrow l = 0$ (as expected!)

2nd order $(b-c)x^2 = 0 \Rightarrow b = c$

3rd order $-dx^3 = 0 \Rightarrow d = 0$.

So $\psi(x) = bx^2 + O(|x|^4)$.

Now recall that

$$R_{a,b}(x) = \pi_c F_{a,b}(x + \psi(x))$$

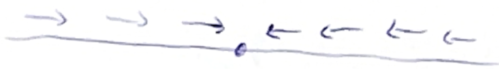
$$= ax^3 + x\psi(x) - x\psi(x)^2 \left(+ O(|x|^4)\right)$$

$$= ax^3 + bx^3 + O(|x|^4)$$

$$= (a+b)x^3 + O(|x|^4)$$

(This is why we assumed $a + b \neq 0$)

We conclude that the origin is unstable if $a+b>0$, as on the center manifold, the origin is not stable:

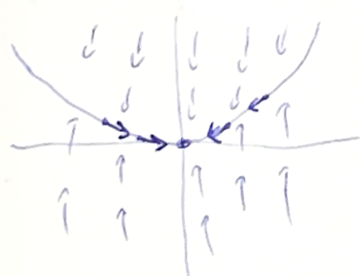


$a+b<0$

$a+b=0$ (needs higher order terms to know more)

$a+b>0$

In fact, we will later see that we may conclude that the origin is stable for $a+b<0$ due to the hyperbolic eigenvalues (just -1 in this case) being stable (having negative real parts)



It should become clear that center-manifold reduction is a very useful technique, but so far we haven't done anything with bifurcations yet. So now, assume we have a family of vectorfields again: $F_\lambda : \mathbb{R}^n \to \mathbb{R}^n$, $\lambda \in \Sigma \subseteq \mathbb{R}^d$

We could get centermanifolds for all systems $\dot{x} = F_\lambda(x)$ individually, but there are two problems:

1) How do we garantee some kind of regularity of the resulting family of center manifolds M_c^λ, $\lambda \in \Sigma$?

2) More importantly, $\dim(M_c^\lambda) = \dim(W_c(DF_\lambda(0)))$ may jump (it will, generically, in a bif. prob.)

(38) Instead, we look at the augmented system on $\mathbb{R}^{n+d}$

$$\begin{aligned} \dot{x} &= F_\lambda(x) \\ \dot{\lambda} &= 0 \end{aligned} \qquad \text{So } \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \tilde{F}(x;\lambda) \text{ where}$$

$$\tilde{F} : \mathbb{R}^{n+d} \circlearrowleft, \qquad \tilde{F}(x;\lambda) = (F_\lambda(x) ; 0)$$

Note that now $\underset{\substack{\uparrow \\ \text{center manifold of } \tilde{F}(x;\lambda)}}{\dim(M_c)} = \dim(W_c(D_{(x,\lambda)}\tilde{F}(0;0)))$

$$\text{where} \quad D_{(x,\lambda)}\tilde{F}(0;0) = \begin{array}{c|c} & \begin{array}{cc} n & d \end{array} \\ \begin{array}{c} n \\ \\ d \end{array} & \left(\begin{array}{c|c} D_x F_\lambda(0;0) & D_\lambda F_\lambda(0;0) \\ \hline 0 & 0 \end{array}\right) \end{array}$$

So

$$\dim(W_c(D_{(x,\lambda)}\tilde{F}(0;0))) = \dim(W_c(D_x F_\lambda(0;0))) + d$$

Note! It is tempting to think

$$W_c(D_{(x,\lambda)}\tilde{F}(0;0)) = W_c(D_x F_\lambda(0)) \oplus \underset{\substack{\nwarrow \\ \{(0;\lambda)\} \subseteq \mathbb{R}^{n+d}}}{\mathbb{R}^d}$$

But this is not always the case!

Consider e.g. $\quad F_\lambda(x,y) = \begin{pmatrix} a\lambda + \mathcal{O}(|(x,y;\lambda)|^2) \\ -y + b\lambda + \mathcal{O}(|(x,y;\lambda)|^2) \end{pmatrix}$

then $D_{(x,y)} F_0(0;0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \rightsquigarrow$ So $\begin{aligned} W_c &= \{(x,0) \mid x \in \mathbb{R}\} \\ W_h &= \{(0,y) \mid y \in \mathbb{R}\} \end{aligned}$

and $D_{(x,y;\lambda)} F_\lambda(0,0;0) = \left(\begin{array}{cc|c} 0 & 0 & a \\ 0 & -1 & b \\ \hline 0 & 0 & 0 \end{array}\right)$

then $W_h = \{(0,y;0) \mid y \in \mathbb{R}\}$ But

$$W_c = \operatorname{span}\left\langle (1,0;0),\ (0,\underset{\substack{\uparrow \\ \text{"a } W_h\text{-component"}}}{\tfrac{1}{b}};1) \right\rangle \quad \text{if } b \neq 0.$$

(39) Usualy, we need an identification

$$W_c(D_{(x;\lambda)}\tilde{F}(0,0;0)) \cong W_c(D_x F_0(0)) \times \mathbb{R}^d$$

But then we get a family of reduced vectorfields:

$$R_\lambda : W_c(D_x F_0(0)) \circlearrowleft$$

Let's explore this a bit more. We write $A = D_x F(0,0)$, $\tilde{A} = D_{(x,\lambda)}\tilde{F}(0,0)$, and $b = D_\lambda F(0,0)$, so that

$$\tilde{A} = \begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{n} & \overbrace{\quad}^{d} \end{array} \\ \left(\begin{array}{c|c} A & b \\ \hline 0 & 0 \end{array}\right) \begin{array}{l} \}\ \mathbb{R}^n \\ \}\ \mathbb{R}^d \end{array} \end{array}$$

as $\mathbb{R}^n$ sits in $\mathbb{R}^n \times \mathbb{R}^d$ through $x \mapsto (x,0)$, we may see $\mathbb{R}^n$ as a subspace of $\mathbb{R}^n \times \mathbb{R}^d$.

Write W_c & W_h for the center- & hyperbolic subspaces induced by A, $\mathbb{R}^n = W_c \oplus W_h$ and $\tilde{W}_c$ & $\tilde{W}_h$ for those induced by $\tilde{A}$, so

$$\mathbb{R}^n \oplus \mathbb{R}^d = \tilde{W}_c \oplus \tilde{W}_h.$$

by identifying $\mathbb{R}^n$ as "$(\mathbb{R}^n, 0)$" in $\mathbb{R}^n \oplus \mathbb{R}^d$, we may see W_c, W_h as subspaces of $\mathbb{R}^n \oplus \mathbb{R}^d$ too.

Then we have:

<u>Lem</u>: in the above setting:

- $W_h = \tilde{W}_h$
- $W_c \subseteq \tilde{W}_c$ and there is a linear map

$$K : \mathbb{R}^d \to W_h \quad \text{s.t.}$$

$$\mathrm{Graph}(K) := \{ (K(\lambda), \lambda) \mid \lambda \in \mathbb{R}^d \} \subseteq \tilde{W}_c$$

and in fact $\tilde{W}_c = W_c \oplus \mathrm{Graph}(K)$

<u>Proof</u>: for $v \in \mathbb{R}^n$,

$$\left(\begin{array}{c|c} A & b \\ \hline 0 & 0 \end{array}\right)\begin{pmatrix} v \\ \hline 0 \end{pmatrix} = \begin{pmatrix} Av \\ 0 \end{pmatrix}$$

Hence $W_c \subseteq \tilde{W}_c$, $W_h \subseteq \tilde{W}_h$. Clearly $\dim(W_h) = \dim(\tilde{W}_h)$ and

$$\dim(W_c) + d = \dim(\tilde{W}_c)$$

So $W_h = \tilde{W}_h$. Now, write $A = \begin{pmatrix} A_c & 0 \\ 0 & A_h \end{pmatrix}$ w.r.t $\mathbb{R}^n = W_c \oplus W_h$

then $A_h : W_h \to W_h$ is invertible, given $\lambda \in \mathbb{R}^d$, set

$K(\lambda) = -A_h^{-1}\pi_h b\lambda \in W_h$, so $K = -A_h^{-1} \circ \pi_h \circ b$

(Note: $b : \mathbb{R}^d \to \mathbb{R}^n$). then we have

$$\tilde{A}\begin{pmatrix} K(\lambda) \\ \lambda \end{pmatrix} = \left(\begin{array}{c|c} A & b \\ \hline 0 & 0 \end{array}\right)\begin{pmatrix} K(\lambda) \\ \lambda \end{pmatrix} = \begin{pmatrix} AK(\lambda) + b\lambda \\ 0 \end{pmatrix} =$$

$$\begin{pmatrix} -AA_h^{-1}\pi_h b\lambda + b\lambda \\ 0 \end{pmatrix} = \begin{pmatrix} -A_h A_h^{-1}\pi_h b\lambda + b\lambda \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} b\lambda - \pi_h b\lambda \\ 0 \end{pmatrix} = \begin{pmatrix} \pi_c b\lambda \\ 0 \end{pmatrix} \in W_c \subseteq \tilde{W}_c$$

Now suppose $X \in \mathbb{R}^{n+d}$ satisfies $\tilde{A}X \in \tilde{W}_c$, then $X \in \tilde{W}_c$

For, write $X = \tilde{\pi}_h X + \tilde{\pi}_c X$, then $\tilde{A}X = \tilde{A}_h\tilde{\pi}_h X + \tilde{A}_c\tilde{\pi}_c X \in \tilde{W}_c$

$\tilde{\pi}_{h/c}$ w.r.t: $\mathbb{R}^{n+d} = \tilde{W}_c + \tilde{W}_h$

so $\tilde{A}_h\tilde{\pi}_h X = 0$. But $\tilde{A}_h$ is invertible, so $\tilde{\pi}_h X = 0$

So $X \in \tilde{W}_c$.

So from before, $\left\{\begin{pmatrix} K(\lambda) \\ \lambda \end{pmatrix} \middle| \lambda \in \mathbb{R}^d\right\} = \mathrm{Graph}(K) \subseteq \tilde{W}_c$

By dimensions, (and transversality, i.e. $v \in W_c \cap \mathrm{Graph}(K) \Rightarrow v = 0$)

we find

$$\tilde{W}_c = W_c \oplus \mathrm{Graph}(K).$$

Cor: Define $\pi_{c/h}$ w.r.t $\mathbb{R}^n = W_c \oplus W_h$, and $\tilde{\pi}_{c/h}$ w.r.t $\mathbb{R}^{n+d} = \tilde{W}_c \oplus \tilde{W}_h$, then

$$\tilde{\pi}_c(x_c + x_h; \lambda) = (x_c + K(\lambda); \lambda) \qquad x_c \in W_c,\ x_h \in W_h,\ \lambda \in \mathbb{R}^d$$

So $\tilde{\pi}_c(x;\lambda) = (\pi_c(x) + K(\lambda); \lambda)$

$\tilde{\pi}_h(x;\lambda) = (\pi_h(x) - K(\lambda); 0)$

(41) Proof Let $\tau(x_c, x_h, \lambda) := (x_c + K(\lambda); \lambda)$, so $\tau : \mathbb{R}^n \oplus \mathbb{R}^d \circlearrowleft$

Then
$$\tau(x_c, 0; 0) = (x_c + K(0); 0) = (x_c, 0, 0)$$
$$\tau(0, K(\lambda); \lambda) = (0, K(\lambda); \lambda) \quad \text{and}$$
$$\tau(0; x_h, 0) = (0, 0, 0)$$

So $\tau|_{\tilde{W}_c} = Id_{\tilde{W}_c}$, $\tau|_{\tilde{W}_h} = 0$. Thus $\tau = \tilde{\pi}_c$

then $\tilde{\pi}_h = Id_{\mathbb{R}^{n+d}} - \tilde{\pi}_c$, so
$$\tilde{\pi}_h(x; \lambda) = (x; \lambda) - (\pi_c(x) + K(\lambda); \lambda) = (\pi_h(x) - K(\lambda); 0) \quad \square$$

Lemma: Define $K : W_c \oplus \mathbb{R}^d \to \tilde{W}_c$
$$(x_c; \lambda) \mapsto (x_c + K(\lambda); \lambda)$$
then K is a linear bijection

and $K^{-1} : \tilde{W}_c \to W_c \oplus \mathbb{R}^d$ is given by
$$(x; \lambda) \mapsto (\pi_c(x); \lambda)$$

Proof: Define $\check{K} : \tilde{W}_c \to W_c \oplus \mathbb{R}^d$ by $(x; \lambda) \mapsto (\pi_c(x); \lambda)$ then
$$K\check{K}(x, \lambda) = K(\pi_c(x); \lambda) = (\pi_c(x) + K(\lambda); \lambda) = (x; \lambda) \text{ as } (x, \lambda) \in \tilde{W}_c$$
$$(\text{so } \pi_h(x) = K(\lambda))$$
$$\check{K}K(x_c; \lambda) = \check{K}(x_c + K(\lambda); \lambda) = (\pi_c(x_c + K(\lambda)); \lambda) = (x_c; \lambda). \quad \square$$
so $\check{K} = K^{-1}$ and both are invertible

Let $\Psi : \tilde{W}_c \to \tilde{W}_h$ be the map definining the centermanifold of the augmented system $\tilde{F}$. Then we get the reduced system $\tilde{R}$ on $\tilde{W}_c$, given by $\tilde{R}(\tilde{x}_c) = \tilde{\pi}_c \tilde{F}(\tilde{x}_c + \Psi(\tilde{x}_c))$, $\tilde{x}_c \in \tilde{W}_c$. Conjugating by the above map K, we get a conjugate system on $W_c \oplus \mathbb{R}^d$, given by
$$\hat{R}(x_c; \lambda) = K^{-1} \tilde{\pi}_c \tilde{F}(x_c + K(\lambda) + \Psi(x_c + K(\lambda); \lambda); \lambda)$$
$$= K^{-1} \tilde{\pi}_c \begin{pmatrix} F(x_c + K(\lambda) + \Psi(x_c + K(\lambda); \lambda); \lambda) \\ 0 \end{pmatrix}$$

Now note that, for $x \in \mathbb{R}^n$,
$$K^{-1} \tilde{\pi}_c(x, 0) = K^{-1}(\pi_c(x) + K(0); 0) = K^{-1}(\pi_c(x); 0) = (\pi_c(x); 0).$$

So
$$\hat{R}(x_c; \lambda) = \begin{pmatrix} \pi_c F(x_c + K(\lambda) + \Psi(x_c + K(\lambda); \lambda); \lambda) \\ 0 \end{pmatrix} \begin{matrix} \leftarrow \mathbb{R}^n \\ \leftarrow \mathbb{R}^d \end{matrix}$$
$$= \begin{pmatrix} R(x_c; \lambda) \\ 0 \end{pmatrix},$$
where $R : W_c \oplus \mathbb{R}^d \to W_c$ is given by
$$R(x_c; \lambda) = \pi_c F(x_c + K(\lambda) + \Psi(x_c + K(\lambda); \lambda); \lambda)$$

(42) So, we now get a parametrized system on W_c, given by

$$\begin{aligned}\dot{x}_c &= R(x_c;\lambda)\\ \dot{\lambda} &= 0\end{aligned}, \quad \text{so} \quad \dot{x}_c = R(x_c;\lambda).$$

Some terms: $R(0;0) = \pi_c F(0 + K(0) + \Psi(0;0);0) = 0$

$$D_{x_c} R(0;0) = \pi_c D_x F(0;0) = \pi_c A = A_c$$

$$D_\lambda R(0;0) = \pi_c D_x F(0;0) K + \pi_c D_\lambda F(0;0)$$

$$= \pi_c A_h \circ K + \pi_c b = \pi_c b$$

as $K: \mathbb{R}^d \to W_h$, and recall $b := D_\lambda F(0;0)$

Note: A_c has its full spectrum on $i\mathbb{R} \subseteq \mathbb{C}$ (by assumption)
But we, in principle, have no assumptions on $D_\lambda R(0;0)$.
Thus, a system such as
$\dot{x}_c = R(x_c;\lambda) = \lambda - x^2 + \mathcal{O}(|x|^3 + |x||\lambda| + |\lambda|^2)$ can appear on the center manifold. In fact, "generically" it will be what appears, (after rescaling) so that the saddle-node can be shown to be the generic bifurcation in 1-parameter, of steady states

Finally, note that all constructions above, in particular of K, K^{-1} and the fact that $W_h = \tilde{W}_h$, guarantee that the whole elaborate conjugation between the system $\dot{x}_c = R(x_c;\lambda)$, $\dot{\lambda} = 0$ and the restriction of $\dot{x} = F(x;\lambda)$, $\dot{\lambda} = 0$ to its center manifold, all respect "λ = constant" slices.

In particular, non-zero values of λ still give flow-invariant ~~subspaces~~ manifolds for the systems

$\dot{x} = F(x;\lambda)$. (λ constant)